INTRODUCTION TO CALCULUS

MATH 1A

Unit 1: What is Calculus?

Lecture

1.1. In this welcome lecture we start with a bit of an overview, what calculus is about and **how it can model the world**. Calculus looks at **changes** from the past, to find **models** and **laws**, which then allow to predict the **future**. The analysis of the past uses **differences**, leading to the concept of **derivative**. Solving the model requires **summation**, leading to the concept of **integral**. **Functions** allow to **model** the situation. Their **graphs** bring in some **geometry** as the derivatives lead to **slope** and **concavity** and the integral leads to **area** or **volume**. The following figure shows example of a function, describing a ball bouncing on floor. Can you describe its concavity features?

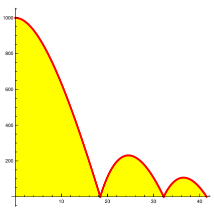

FIGURE 1. The bouncing of a stone is modeled by a function $f(t)$ which tells how high the ball is at time t. We look at $f(t+h)-f(t)$ to see how f **changes** from t to $t+h$. If we have a law for these changes, we can look into the future and predict where the ball will end up. The **domain** of the function are the t for which the function is defined or considered. In this case, it is $t \in [0, \infty)$ as we do not consider negative time.

1.2. How do we analyze a function? Taking differences can help. Functions $f(x)$ are often given just at a set of points. This produces **data points** like $f(1) = 3, f(2) = 9, f(3) = 19, f(4) = 33, f(5) = 51, f(6) = 73, f(7) = 99$? Can you predict the next term? We will discuss this in class.

1.3. In order to gain visual insight, it is helpful to **draw the data**. We call the picture a **graph**. There are many ways on how one can do that and we will collect some ideas in class. The next figure shows one way to **visualize the data**. It is a **bar chart**.

1.4. The **Fibonnacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \dots$ defines a function. We have for example $f(6) = 8$. Can you see the pattern? Can you predict the future and see the next term? Again, if you should not know the rule, look at the rate of change and see whether you can see a rule.

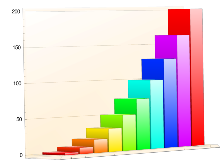

FIGURE 2. When plotting the sequence of numbers in the coordinate plane, the function is visualized as a graph.

HOMEWORK

This problem set is due in grade scope on Wednesday, 1/24/2024 at 9 AM.

Problem 1.1: a) Give a formula of a function $f(x)$ that is decreasing and concave down on the domain $(0, \infty)$.
b) Modify your function in a) so that its graph passes through $(1, 1)$ and $(2, 0)$.

Problem 1.2: Give a formula for an example of a function with
a) domain $(-\infty, \infty)$ such that $2 \leq f(x) \leq 3$.
b) domain $(2, \infty)$ which is decreasing and satisfies $f(3) = 1$. (Make sure the function is not defined on $(-\infty, 2]$).
c) domain $(-\infty, \infty)$ which is always decreasing and is always larger than 1.
d) domain $(-\infty, \infty)$ for which $f(2x) = 4f(x)$.

Problem 1.3: If the Capitol movie theater in Arlington sells a ticket for $x = 10$ dollars, it sells $f(x) = 1000$ tickets. If the ticket prize is increased by 1 to $x = 11$ dollars, the number of tickets decreases by 5. Model the situation with a linear function $f(x) = ax + b$ which predicts the number of tickets sold for prize x.

Problem 1.4: Galileo experimented with a ball on the inclined plane and measured the distances traversed in equal time. He saw the numbers $1, 3, 5, 7, 9, 11, \dots$ and called this the **odd number rule**. Find a function $f(x)$ with the property that $g(x) = f(x+1) - f(x)$ gives the odd numbers $2x + 1$. Describe its concavity nature.

Problem 1.5: Predict the future and find the next term in the sequence

$$2, 10, 30, 68, 130, 222, 350, 520, 738, 1010, 1342, \dots.$$

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Unit 2: Functions

Lecture

2.1. A **function** is a rule which assigns to a real number a new real number. The **domain** of f consists of the points where f is defined and a **codomain** B a set of numbers in which f is mapped to. The **range** is $f(A)$. A function $g(x) = 1/x$ for example can not be evaluated at 0 so that the domain must exclude the point 0. Its range is also $\mathbb{R} \setminus \{0\}$, the set of real numbers without 0. The **inverse** of a function f is a function g such that $g(f(x)) = x$. The function $g(x) = \sqrt{x}$ for example is the inverse of the function $f(x) = x^2$ on its domain $\mathbb{R}^+ = [0, \infty)$. The function $f(x) = 1/x$ is its own inverse on $(-\infty, 0) \cup (0, \infty)$.

identity	x
linear	$3x + 1$
quadratic	x^2
cosine	$\cos(x)$
sine	$\sin(x)$

exponential	$e^x = \exp(x)$
logarithm	$\ln(x) = \log(x)$
square root	$\sqrt{x}$
absolute value	$\|x\|$
bell function	e^{-x^2}

We can build new functions by:

addition	$f(x) + g(x)$
multiplication	$f(x) \cdot g(x)$
division	$f(x)/g(x)$
scaling	$2f(x)$
translation	$f(x + 1)$
composition	$f(g(x))$
inverting	$f^{-1}(x)$

Important classes of functions are

polynomials	$x^2 + 3x + 5$
rational functions	$\frac{x+1}{x^4+1}$
exponentials	e^x, b^x
logarithm	$\ln(x), \log_b(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arctan(x)$
roots	$\sqrt{x}, x^{1/3}$

2.2. The **graph** $\{(x, y) = (x, f(x))\ \}$ allows to **visualize** functions. We can "see a function", when we draw the graph. We will learn to graph in class.

Homework

This homework is due on Friday 1/26, 2024.

Problem 2.1: Model the height $h(t)$ of the sea level at time t in Boston, where the tidal range is 10ft. You can assume that the time between low tide and high tide 2π hours and at time $t = 0$, the sea level $h(t)$ is lowest and equal to 0. Give a concrete function $h(t)$ which does the job.

Problem 2.2: The annual mean atmospheric CO_2 concentration $C(x)$ is observed over time.
a) The concentration was measured to be 300 parts per million in 1974 and 400 parts per million in 2024. Find a linear function $C(x) = ax + b$ which models the situation.
b) Now model the situation as an exponential function $C(x) = ab^x$ so that it fits the data.

Problem 2.3: The graph of the **income distribution** of a country is plotted as a function $L(x)$ of x, the bottom fraction of the population, is called the **Lorenz curve**. For example, if the bottom 30 percent earn 5 percent of the country's income then $L(0.3) = 0.05$. The Lorenz curve is shown below.
a) The **Gini index** is defined to be twice the area between the graph of $f(x) = x$ and the graph of $L(x)$. Use the picture to estimate the Gini index of the US.
b) What would the Gini index be, if everybody would earn the same income? We call this a **perfectly equal income distribution**.
c) How large can the Gini index become theoretically? What would that mean? Draw the situation where the bottom 99 percent earn nothing at all and the top 1 percent earns all the income. This is close to the **perfectly unequal income distribution**.

Problem 2.4: The **20:20 ratio** for a country is defined as the ratio of the country's total income earned by the top 20 percent of the households divided by the country's total income earned by the bottom 20 percent. In other words, it is $L(0.8)/L(0.2)$.
a) Find the 20:20 ratio for the USA using the graph given above.
b) What is the 20:20 ratio for a **perfectly equal country**.
c) What is the 20:20 ratio for a **perfectly unequal country**.

Problem 2.5: a) Plot the function $x \sin(1/x)$. Feel free to use technology if you like. Describe some features of this graph.
b) What would you hear, if a membrane would swing like function $\sin(4000\sqrt{x})$ say from $x = 0$ to $x = 10$? Describe this in words, like "wailing firetruck sirene" or a melody for which the "pitch increases in time".

2.3. Here is the graph for Problem 3 and 4:

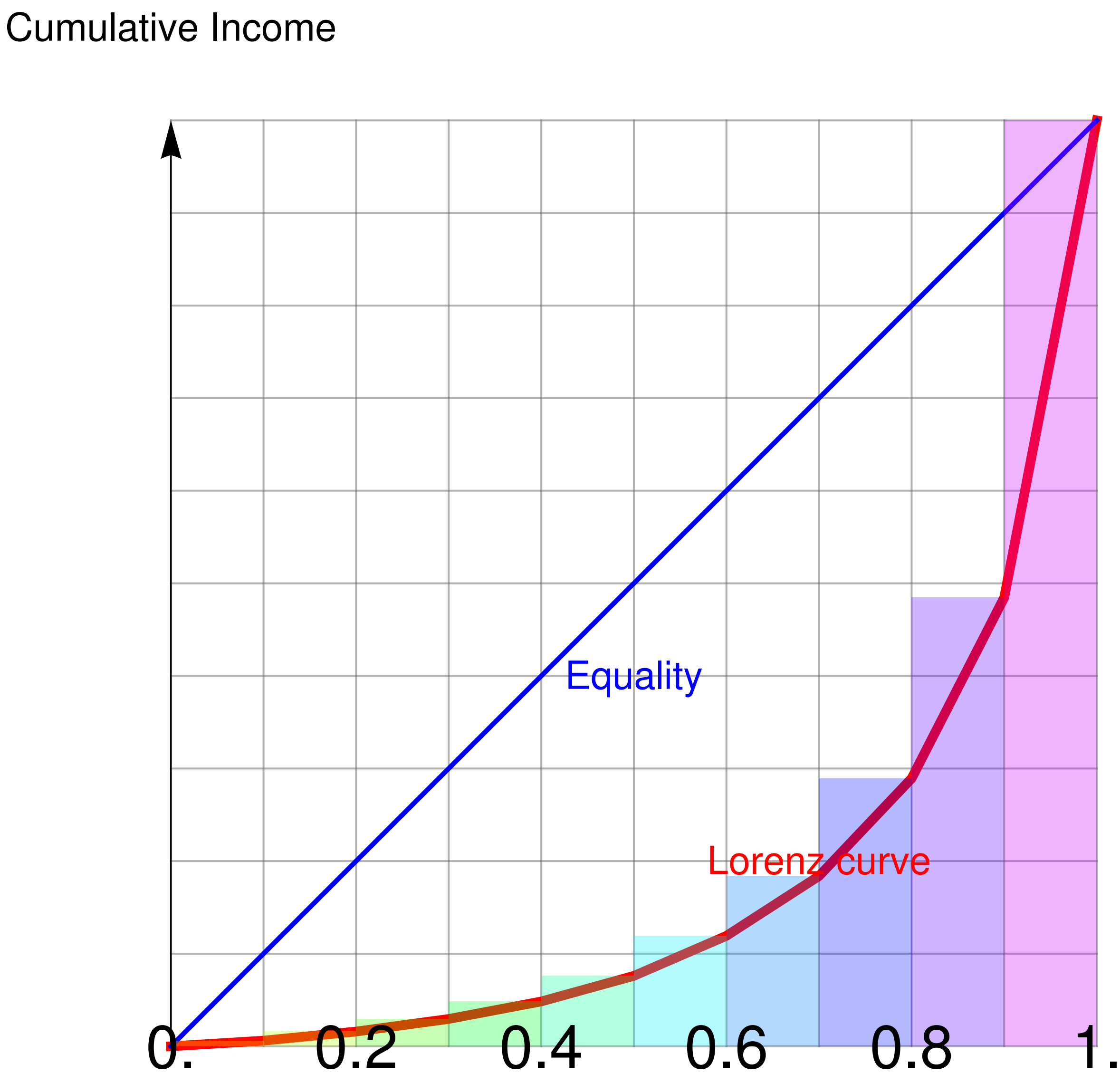

Figure 1. The Lorenz curve.

2.4. The income data points we worked with are:

0	161	260	358	500	730	1142	1707	2784	5161	13613

2.5. This means that the data points for the red curve are (if each square is considered a unit square): (these are the numbers you can work with and which agree with the graph shown above).

0	0.061	0.159	0.295	0.484	0.761	1.193	1.839	2.894	4.847	10.0

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024

Unit 3: Rate of Change

Average rate of change

3.1. The **average rate of change** of a function f on the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a} .$$

The **instantaneous rate of change** at a point x is for now informally defined as the **slope of graph function** at x. The name "average rate of change" will be justified later as it will be identified with the average of all derivatives $f'(x)$ on the interval $[a, b]$. Once we will see the concept of limit, we will identify the **instantaneous rate of change** as a **limit** $[f(x+h) - f(x)]/h$ when h goes to zero. We will get to soon. It is a bit puzzling as when $h = 0$, we have an indeterminate form $0/0$. For this current lecture, it we work with the **slope of the tangent** at x.

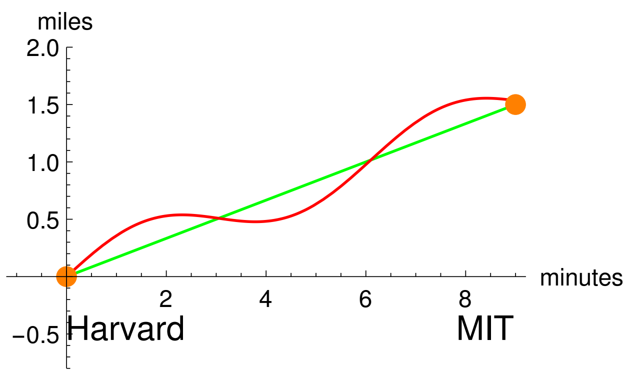

FIGURE 1. A trip can be quantified by the function giving the distance from Harvard as a function of time.

3.2. The average rate of change also makes sense if the function is known only for a few data points. Assume you run the $2km$ from **Harvard to MIT** along the Charles and you clock it with 12 minutes. Then your average speed is $2/12 = 1/6$ km per minute. The instantaneous rate of change varies over the run. You might run slower before the Boston university bridge but then accelerate near the de Wolfe Boathouse, where it goes down. If we measured the progress every second a few times, we would see also that there are variations related to small variations related to getting off and landing on the floor. If you have a smart watch, it measures average rates of changes by looking up your location using **GPS satellites** every once in a while. The smart watch would talk to you during the run and inform you that you ran with $6min/km$.

Instantaneous rate of change

3.3. The **instantaneous rate of change** will later be defined as a limit of average rates of changes when the interval $[x, x+h]$ gets smaller and smaller, but it can also be understood intuitively and geometrically as the **slope of the tangent** at the graph. If you look at the function $f(x) = x^2$, the slope at a point x is $2x$. Lets look at the average rate of change on $[0, h]$ for this function. It is $[f(x+h) - f(x)]/h = [(x+h)^2 - x^2]/h$. If we foil out this expression and simplify, it becomes $[x^2 + 2hx + h^2 - x^2]/h = 2x + h$. One can see that if h goes to zero, the limit $2x$ appears. This value $2x$ what we will call the derivative of f at the point x.

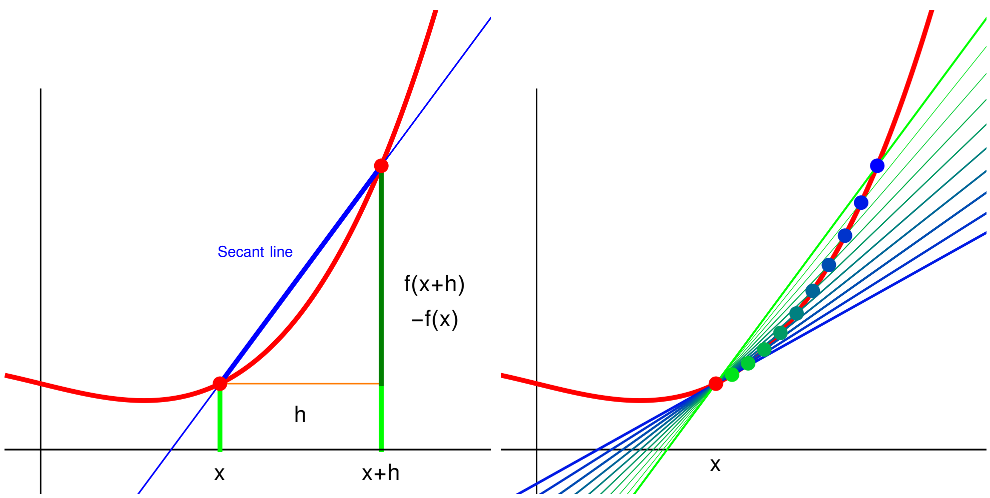

Figure 2. The average rate of change is the slope of the secant connecting two points on the graph. The instantaneous rate of change is a limiting value when the intervals get smaller and smaller. It is for now the slope of the tangent at x.

3.4. Historically, the notion of derivative needed time to develop. One of the first, who investigated the notion seriously was **Zeno of Elea** who was born around 490 BC, just around the time, when Pythagoras (570-495 BC) died. Already Aristotle objected to the paradoxa that Zeno relied on the false supposition that time is composed of indivisible "nows" or "instants". Since we do not know how space and time looks on the **Planck scale** The question of Zeno remains of interest today. [1]

3.5. The nice thing about the average rate of change is that it does not use any notion of limit.

[1]See the book "Zeno's paradox'" (2008) by Joseph Mazur or watch "Ant man and the Wasp"

This homework is due on Monday 2/29, 2024.

Problem 3.1: The 10 meter marks of **Usain Bolt'** during his record run during the Beijing 2008 Olympics are:

Distance	10	20	30	40	50	60	70	80	90	100
Time	1.85	2.87	3.78	4.65	5.50	6.32	7.14	7.96	8.79	9.69

a) What is Bolt's average speed over the entire range?
b) What is Bolt's average speed over the first 50 meters?
c) What is Bolt's average speed over the last 50 meters?
d) Is the average of b) and c) equal to the result in a)?
e) Over which of the 10 meter intervals is the average speed highest?

Problem 3.2: The population of **Pandora** between 2000 and 2024 is modeled by the function $P(t) = 60 + 3\log_2(t+1)$, giving the number of millions of Na'vi people on the planet t years after 2000.
a) What is the average rate of change of the population between $t = 1$ and $t = 7$?
b) Sketch the graph of $P(t)$. From this graph, without computing the numbers order the following quantities in increasing numbers:
A) The average rate of change between $t = 0$ and $t = 4$ B) The average rate of change between $t = 1$ and $t = 4$ C) The average rate of change between $t = 4$ and $t = 9$ D) The instantaneous rate of change at $t = 9$.

Problem 3.3: a) For a given function $f(x)$ denote with $Df(x) = f(x+1) - f(x)$ the average rate of change between x and $x + 1$. Compute this for the function $f(x) = x(x-1)$.
b) Now change the size of the interval and let $Df(x) = (f(x+h) - f(x))/h$ denote the average rate of change between x and $x + h$. Compute this for the function $f(x) = x(x-h)$.
c) Can you see the instantaneous rate of change of $f(x) = x^2$ at a point x?

Problem 3.4: Repeat what you did in problem 3.3 a),b) for the function $f(x) = 2^x$. We can not yet compute the limit $h \to 0$ for the expression you get in c) but just compute it for a very small number like $h = 1/1000$.

Problem 3.5: This is an exploratory question which you definitely need to discuss:
a) Assume you have a function which has the property that the average rate of change between x and $x + 1$ is 0 for all x? Does it mean that $f(x)$ is constant? Or can you find an example of a non-constant function where this happens?
b) Assume you have a function which has the property that the instantaneous rate of change at every point exists and is 0. Does it mean that $f(x)$ is constant? If the answer is no, provide a non-constant function with that property.

Some figures

3.6. Some figures:

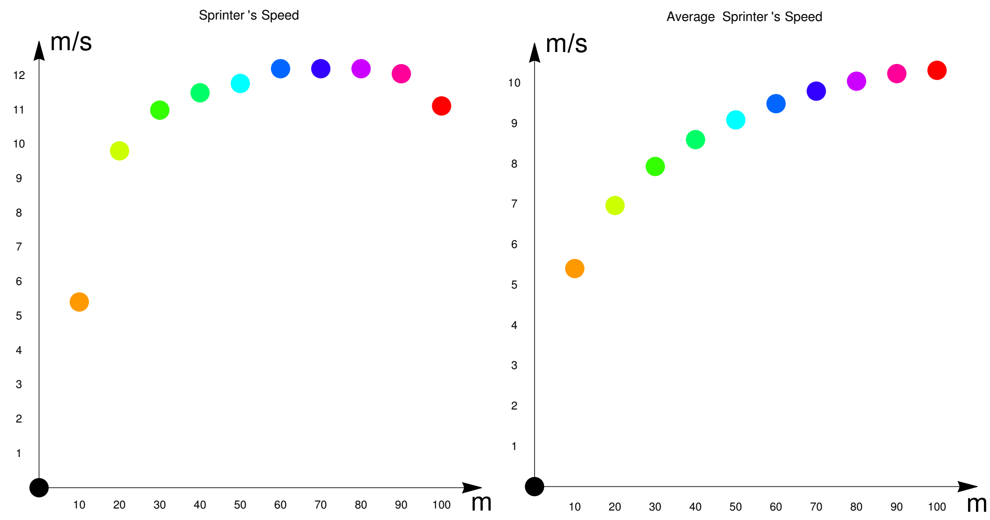

FIGURE 3. The first graph shows the average speed of Bolt on the intervals $[0, 10], [10, 20], \dots, [90, 100]$. This is the **rolling speed** and an approximation for the **instantaneous rate of change**. The second graph shows the **average speed** on the already traversed part $[0, x]$, where x is the number of meters. The last point shows the average rate of change of the run over the entire interval from 0 to 100 m. A speed of 12 m/s corresponds to 43km/h. One year later, in Berlin in 2009, Bolt ran a 9.58 second 100 meter final.

FIGURE 4. Part of the renaissance fresco **"School of Athens"** by Raphael painted around 1510 shows **Zeno of Elea** to the very right. He runs away in terror after realizing that protection from arrows by following his mantra ("There is no motion because at every moment, the arrow is fixed!"), does not seem to work.

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Unit 4: Limits

4.1. We write $x \to a$ to indicate that x **approaches** a. It can approach **from the left** $x \to a^-$ or **from the right** $x \to a^+$. Write $\lim_{x\to a^-} f(x)$ for the **limit from the left** if it exists $\lim_{x\to a^+} f(x)$ for the **limit from the right** if it exists. If both limits exist and agree, we say that $\lim_{x\to a} f(x)$ exists. For $f(x) = \text{sinc}(x) = \sin(x)/x$ for example, the value at $a = 0$ is not defined. To investigate the limit from the right, evaluate $\text{sinc}(0.01) = \sin(0.01)/0.01 = 0.999983$ and $\text{sinc}(0.001) = 0.9999998333$. It looks as if the limit from the right is 1. The function is even, meaning $f(-x) = f(x)$ so that also the limit from the left appears to be 1. The function $\text{sinc}(x) = \sin(x)/x$, used **signal processing** will be discussed more in class.

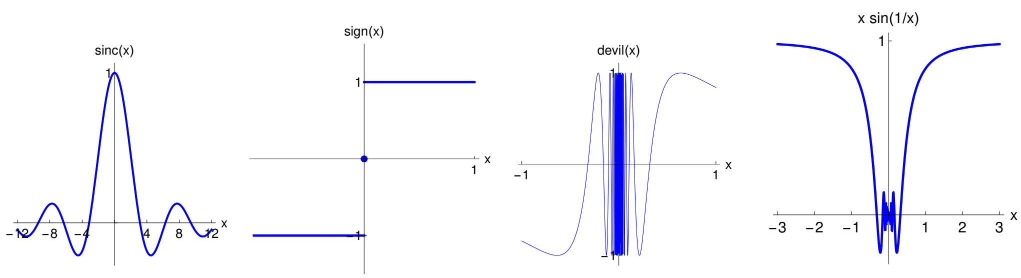


FIGURE 1. The sinc function, the sign function, the devil function $f(x) = \sin(1/x)$ and the **tamed devil function** $f(x) = x\sin(x/x)$.

4.2. $f(x) = \text{sign}(x)$ is defined to be 1 of $x > 0$ and 0 for $x = 0$ and equal to -1 for $x < 0$. In this case, both left and right limits do not exist. Can you modify the function so that the left limit exists and the right limit does not exist?

4.3. For $f(x) = \frac{1-\cos(x)}{x^2}$, it is not so clear what happens at $a = 0$. We can not plug in $x = 0$ because then we divide 0 over 0. But we can evaluate the function for x values close to $x = 0$ and see what happens. Lets see: $\frac{1-\cos(0.1)}{0.1^2} = 0.499583$ $\frac{1-\cos(0.01)}{0.01^2} = 0.499996$. What actually happens is that $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = 1/2$. We will learn how to check that later in the course.

4.4. $f(x) = 1/x^2$ is not defined at $a = 0$. This function has a **pole** at $x = 0$ or a "vertical asymptote".

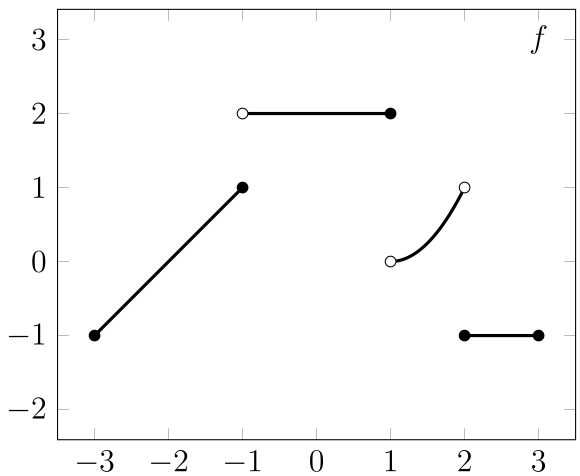
4.5. For the function $f(x) = \sin(1/x)x$ for $x \neq 0$ and $f(0) = 0$ we have $|f(x)| \leq |x|$. The function $f(x)$ converges to 0 for $x \to 0$.

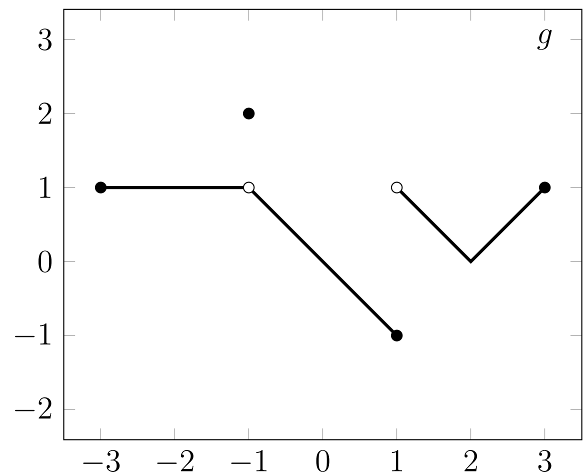
4.6. The function $f(x) = x \log_{10} |x|$ is not defined at $x = 0$. But we can look at small numbers $x > 0$ and small numbers $x < 0$ to investigate the values. We check for example $f(10^{-k}) = \frac{-k}{10^k}$. and $f(-10^{-k}) = \frac{k}{10^k}$. Can you see the limit?

Homework, due Wednesday 1/31/2024

Problem 4.1: a) We do not know the value of $f(x) = (x^4 - 1)/(x - 1)$ at $x = 1$. Nevertheless there is a natural value which can attach to at $x = 1$. Find this value.
b) Use a calculator to find the limit $\lim_{x \to 1} \frac{2^x - 2}{x - 1}$. Make a convincing statement why the left or right limits exist or not.

Problem 4.2: For both f and g below, find the left and right limit at the points $-2, -1, 0, 1, 2$. At $x = 3$ only the left limits, at $x = -3$ only the right limits are needed.





Problem 4.3: Investigate the function $f(x) = \sin(\pi/x)$ at the point $x = 0$. a) First draw the graph of $\sin(\pi/x)$ on the interval $[-2, 2]$. b) Especially evaluate the function at $-2, -1, -1/2, -1/3, -1/4, 1/4, 1/3, 1/2, 1, 2$ and mark them on your graph.

Problem 4.4: a) Draw the graph of a function such that $\lim_{x \to 0^+} f(x) = 1, \lim_{x \to 0^-} f(x) = 2$ and for which $f(0) = 0$. Use the drawing conventions you observe from the previous problem.
b) Find a concrete example of a function for which the limit $\lim_{x \to a}$ does not exist at every point but for which the function is defined at every point. If you are stuck, grab some salt and pepper and season the paper with it. Then eat it.

Problem 4.5: Explore the function $f(x) = x^x$ at $x = 0$. Make experiments. Can you say something about the limit from the right or limit from the left?

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Unit 5: Continuity

5.1. A **function** f is called **continuous** at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \to f(x_0)$ for $x \to x_0$. A function f is **continuous on the interval** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$. This means intuitively, we can **draw the graph of the function without lifting the pencil**. In (a, b), the limit needs to exist both from the right and from the left. Continuity means that small changes in x results in small changes of $f(x)$. Any polynomial like $x^3 + 2x - 4$ or trig functions like $\cos(x), \sin(x)$ or exponential functions $\exp(x)$ are continuous.

Rules:
a) If f and g are continuous, then $f + g$ is continuous.
b) If f and g are continuous, then $f * g$ is continuous.
c) If f and g are continuous and if $g \neq 0$ everywhere, then f/g is continuous.
d) If f and g are continuous, then $f \circ g(x) = f(g(x))$ is continuous.

5.2. The **squeeze theorem** is a tool to check continuity at a point a.

If $g(x) \leq f(x) \leq h(x)$ for functions g, h continuous at a and $g(a) = h(a) = b$, then f is continuous at a and $f(a) = b$.

The reason is that $|h(x) - g(x)| \to 0$ and $g(x) \to b$ and $|f(x) - b| \leq |h(x) - g(x)| + |g(x) - b| \leq |h(x) - g(x)| + |g(x) - b| \to 0$ so that also $f(x) \to b$.

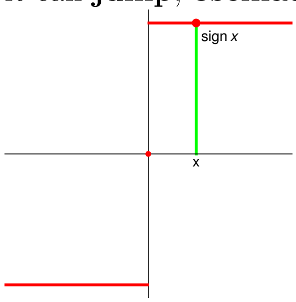
5.3. The function $f(x) = 1/x$ is continuous except at $x = 0$. There is **pole discontinuity** at $x = 0$. The graph has a **vertical asymptote**.

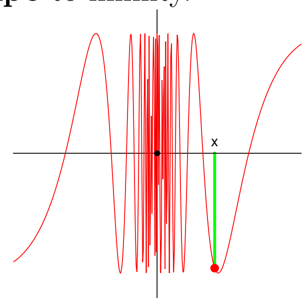
5.4. The logarithm function $f(x) = \ln|x|$ is continuous for all $x \neq 0$. It can not be fixed $x = 0$ because $f(x) \to -\infty$ for $|x| \to 0$.

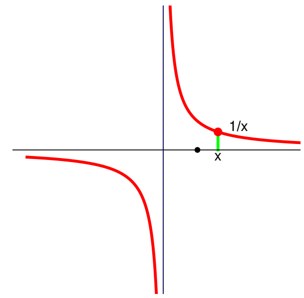
5.5. The **co-secant function** $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi$ and any multiple of π.

5.6. The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It fails continuity because of **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just pick $x_n = 2/(4k + 1)$ or $z_n = 2/(4k - 1)$.

5.7. There are three major reasons, why a function can be not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity.







Homework, Due Friday 2/2/2024

Problem 5.1: a) Define $f(x) = x^2 \cos(1/x)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Use the squeeze theorem to see that this function is continuous everywhere.
b) Define $f(x) = \cos(1/x)$ for $x \neq 0$ and $f(x) = 1$ for $x = 0$. Verify that this function is not continuous at 0.

Problem 5.2: The number of users (in millions) of a social network is modeled as a function that is linear $U(t) = at + b$ for $t \geq 5$ and exponential $U(t) = ce^{kt}$ for $t \in [0, 5]$. Assume that we have 2 million users initially $U(0) = 2$. Data fitting leads to $a = 220$ and $b = 380$ for the linear growth. Determine k so that the function is continuous at 5.

Problem 5.3: a) The function $f(x) = (e^{2x} - 1)/(e^x - 1)$ is not defined at $x = 0$. Can you find a value $f(0) = b$ so that the function is continuous everywhere?
b) The function $f(x) = (x^3 + 2x^2 - 2x - 1)/(x - 1)$ is not defined at $x = 1$. Can you find a value $f(1) = b$ so that with this postulate the function is continuous?

Problem 5.4: Which of the following functions are continuous everywhere?
a) $f(x) = \text{sign}(x) + \sin(1/x)$
b) $f(x) = x\text{sign}(x) + \sin(1/x)$
c) $f(x) = \text{sign}(x) + x \sin(1/x)$
d) $f(x) = x\text{sign}(x) + x \sin(1/x)$

Problem 5.5: Which functions can be made continuous by "fixing broken places" (assign a value to an initially not defined point)? If the function can be fixed show how.
a) $\text{sinc}(5x) + \sin(x)/(2 + \sin^2(x)) + (x^3 - 1)/(x - 1)$, b) $\sin(\tan(x))$
c) $\tan(\sin(x)) + \frac{x^2+5x+x^4}{x-3}$
d) $\tan(2\sin(x)) + \frac{x^2+5x+x^4}{x-3}$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 6: Derivative

6.1. The **derivative** of a function $f(x)$ at a point x is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists. It is the **instantaneous rate of change** we introduced earlier: it is the slope of the tangent at the point x.

6.2. Let us look at the function $f(x) = x^2$. We have

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{2hx + h^2}{h} .$$

For $h \neq 0$, we can divide by h and equate this to $2x + h$. We can now take the limit $h \to 0$ and see that $f'(x) = 2x$.

6.3. We will derive in class that in general, the function $f(x) = x^n$ leads to

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h} = \frac{nhx + \dots}{h}$$

which similarly as before simplifies to $nx^{n-1} + h(R(x))$, where $R(x)$ is a polynomial. Now again, for $h \to 0$, have

$$\boxed{\tfrac{d}{dx} x^n = nx^{n-1}}$$

6.4. For the **exponential function** $f(x) = e^x$, one can also compute the derivative

$$\frac{e^{x+h} - e^x}{h} = \frac{[e^x e^h - e^x]}{h} = e^x \frac{e^h - 1}{h} .$$

Now we can see that the limit $(e^h - 1)/h$ goes to 1 as $h \to 0$. Proving this depends on how the exponential function is defined. A calculator for example implements the exponential function as $e^h = 1 + h + h^2/2 + h^3/6 + \cdots$ from which you can see that $e^h - 1$ is divisible by h. Taking the limit $h \to 0$ gives then 1.

$$\boxed{\tfrac{d}{dx} e^x = e^x}$$

Homework

This homework is due on Monday 2/05/2024.

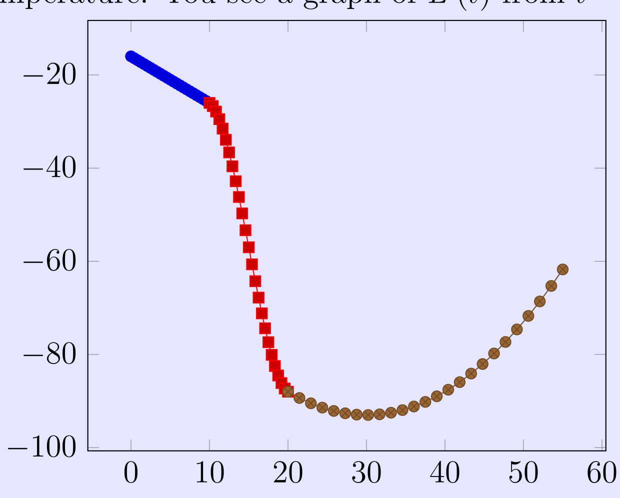
Problem 6.1: a) Find the limit $\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$ at the point $x=2$.
b) Do the same with the function x^3-x instead of x^3.

Problem 6.2: a) Use the definition of the derivative to compute the derivative of $(x+1)/(x-2)$ at $x=0$.
b) Compute the limit $\lim_{h\to 0} \frac{\sqrt{4+h}-\sqrt{4}}{h}$.

Problem 6.3: Give in each case a function with the property or explain why it does not exist:
a) f is continuous at $x=3$ but not differentiable at $x=3$.
b) f is continuous at $x=3$ and differentiable at $x=3$.
c) f is differentiable at $x=3$ but not continuous at $x=3$.

Problem 6.4: Find the limit $f'(x) = \lim_{h\to 0}[f(x+h)-f(x)]/h$. for $f(x)=1/x$.

Problem 6.5: In this QRD problem we want to see what f' tells about f. The water level $L(t)$ of the **Aral sea** dropped between 1960 and 1970 by 21 cm/year, then from 1970-1980, it decreased by 57 cm per year. Afterwards until 2015 (t=50) the drop in water level started accelerating, due to positive feedback between evaporation and Sea Surface Temperature. You see a graph of $L'(t)$ from $t=0$ to $t=55$.



Assume that the water level was 54 m in 1960 (t=0), draw a qualitative picture on how $L(t)$ looks like.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024

Unit 7: Basic Derivatives

7.1. We have already seen the key power identity:

$$\frac{d}{dx}x^n = nx^{n-1} .$$

7.2. As a general rule we can state already now is that for any constant c

$$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

In short $(cf)' = cf'$. The reason is that for every h, we have the property that the average rate of change $(f(x+h) - f(x))/h$ has the property that **it plays nice with linearity**. This property goes over to the limit when the Planck constant h goes to zero.

7.3. An other general rule is the **addition rule**:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

7.4. Now we can compute the derivative of **any polynomial**: for example: lets compute the derivative of $4x^3 + 3x^2 + x + 3$. The answer is $12x^2 + 6x + 1$.

7.5. We have defined $f(x) = e^x$ as the compound interest limit $(1+h)^{x/h}$ for $h \to 0$. [1] Since we can check that $(f(x+h) - f(x))/h = f(x)$ for any $h > 0$, the exponential function also has the property $\frac{d}{dx}e^x = e^x$ and more generally

$$\frac{d}{dx}e^{cx} = ce^{cx}.$$

[1] This limit e^x exists: use the squeeze theorem for $g(x) \leq f(x) \leq h(x)$ with the decreasing $g(h) = (1+h)^{x/h}$ and increasing $k(h) = (1+h)^{x/h}(1+h)$ using that for $h \leq 1$ one has $k(h) - g(h) = h(1+h)^{x/h} \leq h2^x$ which converges to 0 for $h \to 0$.

7.6. If you increase x by a factor c faster, also the slope gets scaled by c. In general

$$f'(cx) = cf(cx)$$

You generalize this slightly in the homework. Later we will learn it as a special case of the chain rule.

7.7. In order to see the derivatives of the trig functions, remember first the **fundamental theorem of trigonometry** and companion identity:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \ , \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0 \ .$$

Both these limits follow from the squeeze theorem. (See the short video). If we divide one of the addition formulas for trig functions by h

$$\sin(x+h) - \sin(x) = \cos(x)\sin(h) + \sin(x)(\cos(h)-1)$$

we get $\cos(x)$ in the limit $h \to 0$. If we divide the second addition formula

$$\cos(x+h) - \cos(x) = \cos(x)(\cos(h)-1) - \sin(x)\sin(h)$$

by h and take the limit we get $-\sin(x)$. We have shown

$$\frac{d}{dx}\sin(x) = \cos(x), \ \frac{d}{dx}\cos(x) = -\sin(x)$$

7.8. In the homework for today you have shown from the definition:

$$\lim_{h\to 0}[\frac{1}{x+h} - \frac{1}{x}]/h = -\frac{1}{x^2} \ .$$

You have also seen in the homework that

$$\lim_{h\to 0}[\sqrt{x+h} - \sqrt{x}]/h = \frac{1}{2\sqrt{x}} \ .$$

These are special cases for the following formula for x^n. Indeed, for **any real number** n also negative ones, we have

$$\frac{d}{dx}x^n = nx^{n-1}.$$

7.9. By the way, we can see this also from writing $x^n = e^{n\ln(x)}$, using the chain rule (covered later in the course) and using $\frac{d}{dx}\ln(x) = 1/x$ which we are going to look at next.

7.10. In the last worksheet, we simplified $\lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h}$ to $\lim_{h\to 0}\frac{\ln(1+h/x)}{h}$. We need to get the limit $\ln(1+h)/h$. Since the limit $h \to 0$ of $(1+h)^{1/h}$ gives e by definition, $\ln(1+h)/h \to 1$ and so $\ln(1+h/x)/h \to 1/x$.

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Homework

This homework is due on Wednesay 2/7, 2024.

Problem 7.1: Compute the following derivatives.
a) $2x^5 + 3x^2 + 4x + 8$.
b) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$.
c) $1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$
d) $\sqrt{x} + x^{3/2} + x^{5/2}$
e) $(1 + x)(1 + x + x^2 + x^3 + x^4)$

Problem 7.2: We have seen that $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$.
a) Why is $\frac{d}{dx}\frac{1}{x-5} = -\frac{1}{(x-5)^2}$?
b) Formulate a general rule $\frac{d}{dx}f(x-a) = f'(..........)$ which holds for any differentiable function f.
c) Formulate a general rule $\frac{d}{dx}f(cx+b) = f'(.........)$ which holds for any differentiable function f.

Problem 7.3: Compute the following derivatives
a) $4\sin(3x) + 7\cos(9x)$
b) $\ln(x) + \ln(2x) + \ln(3x)$
c) $10e^{11x} + 8e^{20x} - 20e^{100x}$
d) $9x^2 + \frac{1}{x^7} + 2\cos(3x) + \ln(7x) - e^x$.
e) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$.

Problem 7.4: a) Compute the derivative of $f(x) = \pi^x$. You first might have to rewrite the function in a form which allows you to use rules you know.
b) What is the derivative of a^x in general if $a > 0$ is an arbitrary number?

Problem 7.5: a) Compute the derivative of $f(x) = \sqrt{3x+5}$ from the rules you know.
b) In order to appreciate what we have achieved, compute the limit

$$\lim_{h\to 0}[f(x+h) - f(x)]/h \ .$$

for the function $f(x) = \sqrt{3x+5}$ the old way as in PSet 6.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024

INTRODUCTION TO CALCULUS

MATH 1A

Unit 8: Derivative Rules

8.1. You have all already used linearity of the derivative. If we multiply a function by a constant c, then the average rate of change $(f(x+h)-f(x))/h$ also gets multiplied by c. We can pass to the limit and see

$$(cf)' = cf'$$

8.2. Also, if we take the sum of two functions $f+g$, this is a new function, whose derivative is the sum of the derivatives of f and g

$$(f+g)' = f' + g'$$

The two properties together show that the process of going from f to f' is linear.

8.3. The product rule for differentiation follows from the identity

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h)-f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h)-g(x)] \ .$$

When dividing by h we get on the left hand side the average rate of change of fg on $[x, x+h]$ and on the right the average rate of change of f times $g(x+h)$ plus f times the average rate of change of g. For $h \to 0$, this is

$$(fg)' = f'g + fg'$$

Figure 1. The product rule. Leibniz.

8.4. The quotient rule allows to differentiate $f(x)/g(x)$ if $g(x) \neq 0$:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[g(x)f'(x) - f(x)g'(x)]}{g^2(x)}$$

"High d low take low d high. Cross the line and square the low." For example, we can see that $\tan'(x) = 1/\cos^2(x)$.

Homework, due 2/9/2024

Problem 7.1: Compute the following derivatives using the product rule:
a) $\frac{x^2}{2} - \frac{2}{x^2}$
b) $e^{5\ln(x)+\ln(2)}$ first rewrite this
c) $(t + \frac{1}{t})(t - \frac{1}{t})$. Use the product rule. Smarty pants solution only additional.
d) $\sin(x)\cos(x)$
e) $\ln(x)e^x \sin(x)$

Problem 7.2: Now compute the following derivatives using the quotient rule:
a) $\cot(x)$
b) $\frac{\ln(x)}{\ln(2x)}$
c) $\frac{x^2+2x+1}{x\sqrt{x}}$
d) $\frac{1+x}{1-x}$
e) $\frac{x^2-1}{x-1}$ Use the quotient rule. Smarty pants solution only additional.

Problem 7.3: Compute the first two derivatives of the cotangent function by hand. This involves both the product and quotient rule. For c)-e) you can make use of computer assistance, if you like. a) $\cot'(x)$, b) $\cot''(x)$, c) $\cot'''(x)$, d) $\cot''''(x)$, e) $\cot'''''(x)$.

Problem 7.4: We break here the Guinness record of the **most sophisticated differentiation problem** ever posed in a college calculus course. First define $f_0(x) = x$, then $f_1(x) = 1/(1+x), f_2(x) = 1/(1+1/(1+x))$ with the rule $f_n(x) = 1/(1+f_{n-1}(x))$. We ask you to compute the derivative of f_{50}! Hint: We actually have $f_n(x) = \frac{F(n)+F(n-1)x}{F(n+1)+F(n)x}$, where $F(n)$ is the n'th Fibonacci number. We give you $F(50) = 12586269025, F(49) = 7778742049$.

Problem 7.5: We test ChatGPT and ask it to differentiate $\sin(x)\cos(x)\tan(x)\log(x)\exp(x)$. It gave us the answer $(x\log(x)\sin(2x) - x\log(x)\cos(2x)/2 + x\log(x)/2 - \cos(2x)/2 + 1/2)\exp(x)/x$. As an AI researcher we want to find out whether this is correct. AI for example failed miserably to solve problem 7.4. Give at least two strategies to verify or falsify the output. You are allowed to use any tools, also AI ... What is the analysis of your expert opinion?

P.S. Here is the Python code which Chat GPT suggested to answer problem 5):

```python
from sympy import sin, cos, tan, log, exp, symbols, diff
x=symbols('x')
f=sin(x)*cos(x)*tan(x)*log(x)*exp(x)
f_prime = diff(f,x)
print(f_prime.simplify())
```

P.P.S. Mathematica gave us for f_{10}, f_{20} (not yet differentiated) for problem 4:

$$\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}$$

$$\frac{1}{x+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}$$

And here is f_{50}, our Guinness book submission entry: (isn't it gorgeous?)

$$\frac{1}{x+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}+1}$$

How do machines solve in this problem? We asked Mathematica to compute f_8'.

$$\frac{1}{(x+1)^2\left(\frac{1}{x+1}+1\right)^2\left(\frac{1}{\frac{1}{x+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1}+1}+1\right)^2\left(\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{\frac{1}{x+1}+1}+1}+1}+1}+1}+1}+1\right)^2}$$

The program gave the output *Up yours*! when asked to give a LaTeX output for f_{50}'.

OLIVER KNILL, KNILL@MATH.HARVARD.EDU, MATH 1A, SPRING, 2024

Unit 9: Hospital

9.1. **Hospital's rule** is a fantastic tool. It allows to compute limits. [1] It is a miracle procedure and the answer to all our prayers to save us from dreadful limit computations!

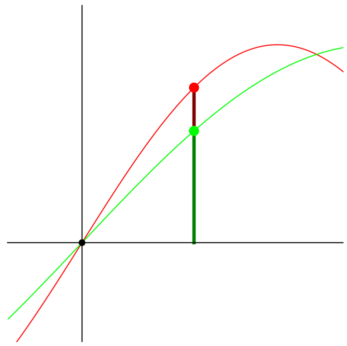
Hospital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then
$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} .$$

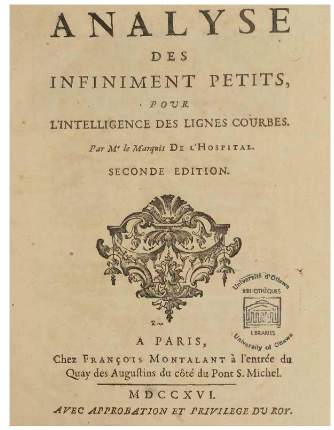
Lets see how it works in examples:

The fundamental theorem of trigonometry:
$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 .$$
Note that this does not replace the derivation because it is equivalent to $\sin' = \cos$!

9.2. The proof of the rule is very simple: since $f(p) = g(p) = 0$ we have for the average rate of changes $Df(p) = (f(p+h) - f(p))/h = f(p+h)/h$ and $Dg(p) = (g(p+h) - g(p))/h = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$. So, the **quantum l'Hospital rule** holds:
$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} .$$
Now take the limit $h \to 0$. On the left we get $\lim_{h \to p} f(x)/g(x)$ by definition. On the right we get $f'(p)/g'(p)$ by definition. Voilà!

[1] Hospital is is easier to write and remember than Hôpital. Bring f to the hospital!





Problem. Find the limit $f(x) = (\exp(2x) - 1)/x$ for $x \to 0$.
Answer. The rule gives 2.

Problem. Find the limit $f(x) = \sin(100x)/\sin(101x)$ for $x \to 0$.
Answer. The rule 100/101.

9.3. The “first calculus book” was “Analyse des Infiniment Petits pour l”intelligence des Lignes Courbes” appeared in 1696. It was written by **Guillaume de l’Hospital** and has about 50-100 pages. [2] The mathematical content is mostly due to **Johannes Bernoulli**. The book remained the standard for a century.

9.4. Sometimes, we have to administer l’Hospital twice:

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f''(x)}{g''(x)}$ if $g''(p) \neq 0$.

Problem: What do you get if you apply l’Hospital to the limit $[f(x + h) - f(x)]/h$ as $h \to 0$?
Answer: Differentiate both sides with respect to h! And then feel awesome!

What is the limit $\lim_{x \to 0} |x|^x$? This will provide the best answer to the question $\boxed{\textbf{What is } 0^0\textbf{?}}$

Find the limit $\lim_{x \to 2} \frac{x^2 - 4x + 4}{\sin^2(x - 2)}$.
Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.

[2] Stewart’s book with 1200 pages probably contains about 4 million characters, about 12 times more than l’Hospital’s book. The OCR text of l’Hospital’s book of 200 pages has 300’000 characters.

Homework

Problem 9.1: For the following functions, find the limits as $x \to 0$ using Hospital:
a) $\sin(7x)/(5x)$
b) $(\exp(16x) - 1)/(\exp(17x) - 1)$
c) $\sin^2(8x)/\sin^2(5x)$
d) $\frac{\tan(4x)}{3x}$
e) $\sin(\sin(11x))/x$.

Problem 9.2: **Luna**, a new math chatbot, teaches itself limits but still makes mistakes and struggles with concepts. Please evaluate its answers to the right:

Problem	Luna's Reasoning
a) $\lim_{x\to\pi} \frac{\cos x}{x-\pi}$	$\lim_{x\to\pi} \frac{\cos x}{x-\pi}$ is by Hospital equal to $\lim_{x\to\pi} \frac{-\sin x}{1} = 0$
b) $\lim_{x\to5} \frac{x-5}{\sqrt{x-5}} = 0$	Hospital: $\lim_{x\to5} 1/(1/(2\sqrt{x-5})) = \lim_{x\to5} 2\sqrt{x-5} = 0$
c) $\lim_{x\to0} \frac{e^x-1}{x}$	$\lim_{x\to0}(e^x - 1) = 0$ and 0 over anything is 0, the limit is 0.
d) $\lim_{x\to0} \ln\|x\|x = 0$	$\ln\|x\| \to -\infty$ and $x \to 0$. As it is not f/g, the limit DNE.
e) $\lim_{x\to0} \ln\|2x\|/\ln\|x\|$	Hospital gives $\lim_{x\to0}(1/2x)/(1/x) = \lim_{x\to0} 1/2 = 1/2$.

Problem 9.3: Use l'Hospital to compute the following limits $x \to 0$:
a) $\lim_{x\to0} x/\ln|x|$
b) $\ln|5x|/\ln|x|$.
c) $4\text{sinc}'(x) = 4(\cos(x)x - \sin(x))/x^2$
d) $\ln|1 + x|/\ln|2 + x|$.
e) $(e^x - 1)/(e^{2x} - 1)$

Problem 9.4: We have seen how to compute limits with healing. Fix the broken bones by bringing them to the Hospital at $x \to 1$:
a) $\frac{x^{100}-1}{x^{22}-1}$.
b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

Problem 9.5: These problems need to be done during commercial breaks of the Super Bowl! If you fail to do so, you will be sent to the hospital by fierce 1a minions.
a) Find the limit $\lim_{x\to0} \frac{x}{\tan(6x)}$.
b) Find the limit $\lim_{x\to5} \frac{x^2-25}{x-5}$
c) Find the limit $\lim_{x\to0} \frac{1-e^x}{x-x^3}$.
d) Find the limit $\lim_{x\to0} \frac{\ln(1+9x)}{4x}$.
e) Find the limit $\lim_{x\to1}(x^7 - 1)/(x^3 - 1)$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 10: Infinity

10.1. This lecture is about infinity. The main point is that Hospital's rule for the **indefinite form** "0/0" works also for the **indefinite form** "∞/∞" as well as when $p = \infty$. Given a function $f(x)$, we can look how $f(x)$ grows when $x \to \infty$. If there is a limit for $x \to \infty$, we have a **horizontal asymptote**. For example $\lim_{x\to\infty} \arctan(x) = \pi/2$. We can also reach infinity vertically. If $\lim_{x\to p} f(x)$ does not exist, there might be a **vertical asymptote**. The function $f(x) = \frac{x^2+1}{x^2-1}$ for example has a horizontal asymptote $y = 1$ as l'Hospital gives $\lim_{x\to\infty} f(x) = 1$. and vertical asymptotes at $x = 1$ and $x = -1$. If $\lim_{x\to p} f(x) = \infty$ and $\lim_{x\to p} g(x) = \infty$ we can ask what happens with the limit of $\lim_{x\to p} f(x)/g(x)$. Again, this can be done with Hospital.

Hospital's rule. If f, g are differentiable and $\lim_{x\to p} f(p) = \lim_{x\to p} g(p) = \infty$ and $\lim_{x\to p} g'(x) \neq 0$, then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \lim_{x\to p} \frac{f'(x)}{g'(x)} .$$

Find $\lim_{x\to\infty}(7x^2 + x + 1)/(3x^2 - 1)$. **Solution.** We check to have an indefinite form ∞/∞. Differentiate both nominator and denominator to get $\lim_{x\to\infty}(14x + 1)/6x$. Having again an indefinite form ∞/∞, we send it again to the Hospital. The answer is $14/6 = 7/3$.

Find $\lim_{x\to 0} \log|7x|/\log|3x|$. This is an indefinite form ∞/∞. We can use l'Hospital and see $\lim_{x\to 0}(7/7x)/(3/3x)$ which can be simplified to 1 for $x \to 0$.

10.2. About the proof: the case when both sides converge to infinity can be reduced to the 0/0 case by writing $A = f/g = (1/g(x))/(1/f(x))$. Use l'Hospital and take the derivative on both sides simplifies to $(g'/f')A^2$. Solving for A gives $A = f'(p)/g'(p)$.

Problem: Lets look at the limit $\lim_{x\to\pi/2} \tan(3x)/\tan(7x)$. First check this is an indefinite form ∞/∞. Now take the derivatives on both sides: $\lim_{x\to\pi/2}(3/\cos^2(3x))/(7/\cos^2(7x)) = 3/7$.

Homework

Problem 10.1: Lets look at the functions $f(x) = \ln(x)$ and $g(x) = x$. In order to see which function grows faster, we study the limit $\lim_{x\to\infty} f(x)/g(x)$.
a) What is $f(x)/g(x)$ for $x = e^{10}$ and $x = e^{1000}$?
b) Compute the limit $\lim_{x\to\infty} f(x)/g(x)$ for $x \to \infty$ using l'Hospital.

Problem 10.2: Let us introduce the notation $f(x) \ll g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 0$. The meaning is that $f(x)$ **grows asymptotically slower** than $g(x)$. For example, $\sqrt{x}\sin(x) \ll x$ because $\sqrt{x}\sin(x)/x \to 0$ for $x \to \infty$.
a) Rank the functions $e^x, x^x, \ln(x), \sqrt{x}, x, x^2$ with that order notation. Which one grows slowest, which is next etc, until which is grows fastest?
b) Produce one single graphics that shows the graphs of all these 6 functions, plotted on the interval $[0, 3]$.

Problem 10.3: Find the following limits involving the indefinite form ∞/∞:
a) $\lim_{x\to 0} \cot(x)/\cot(3x)$.
b) $\lim_{x\to\infty} \frac{3x^2+1}{4x^2+100}$.
c) $\lim_{x\to 0} \sqrt{\log(3x)}/\sqrt{\log(2x)}$.
d) Find $\lim_{x\to\infty}(x^2 + x - 1)/\sqrt{5x^4 + 1}$.
Hint to to c) and d): First square the expression and find the limit, then take the root of the result.

Problem 10.4: Use l'Hospital to compute the following routine limits $x \to \infty$ (we use for a change $\log = \ln$, which is the common notation in all computer programming languages and all higher mathematics).
a) $\log|x|/x$
b) $\log|5x|/\log|x|$.
c) $x^2/(1 + x^2)$.
d) $\log|1 + x|/\log|2 + x|$.
e) $(e^x - 1)/(e^{2x} - 1)$

Problem 10.5: We have $nx = x + x + x + \cdots + x$ and $x^n = x * x * x * \cdots * x$. In computer science, the Knuth arrow notation write this as $x \uparrow n$. The number 3^4 for example is $3 * 3 * 3 * 3 = 256$. Knuth then writes $x \uparrow\uparrow n = x \uparrow x \uparrow \ldots \uparrow x$ meaning to exponentiate n times. For example, $x \uparrow\uparrow 4 = x^{x^{x^x}}$.
a) We have already studied $f(x) = x^x = x \uparrow\uparrow 2$. Lets look at $g(x) = x \uparrow\uparrow 3 = x^{(x^x)}$. Compute the numbers $g(1), g(2), g(3)$ and if you have the tools, compute $g(4)$.
b) What is the limit $\lim_{x\to\infty} f(x)/g(x) = x^x/x^{(x^x)}$? You can use l'Hospital, but you need an idea.

Here are the first $x \uparrow\uparrow n$ for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. We don't write the brackets. The meaning is always that you put the brackets from the right, like $x^{x^x} = x^{(x^x)}$:

$$\left\{ x, x^x, x^{x^x}, x^{x^{x^x}}, x^{x^{x^{x^x}}}, x^{x^{x^{x^{x^x}}}}, x^{x^{x^{x^{x^{x^x}}}}}, x^{x^{x^{x^{x^{x^{x^x}}}}}}, x^{x^{x^{x^{x^{x^{x^{x^x}}}}}}}, x^{x^{x^{x^{x^{x^{x^{x^{x^x}}}}}}}} \right\}$$

And here is $x \uparrow\uparrow 100$.

$$x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x^{x}}}$$

And now lets get dizzy: Knuth defines $x \uparrow\uparrow\uparrow n$ as $x \uparrow\uparrow x \uparrow\uparrow x \dots \uparrow\uparrow x$. For example, $x \uparrow\uparrow\uparrow 2 = x \uparrow\uparrow x$.

Now

$$10 \uparrow\uparrow\uparrow 2 = 10 \uparrow\uparrow 10 = 10^{10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}}$$

There is no way that we can even write down this number. We estimate to have 10^{80} elementary particles in the universe.

Now wrap your head around $10 \uparrow\uparrow\uparrow\uparrow 2 = 10 \uparrow\uparrow\uparrow 10 = 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10 \uparrow\uparrow 10$.
We do not have even to go to infinity to become insane. Even large finite numbers can drive you nuts. Big time.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024

INTRODUCTION TO CALCULUS

MATH 1A

Unit 11: Linearization

11.1. A differentiable function $f(x)$ can near a point a be approximated by

$$L(x) = f(a) + f'(a)(x - a) \ .$$

We call L the **linearization** of f near a. Why is L close to f near a? First of all, $L(a) = f(a)$. Next, we check that $L'(a) = f'(a)$. The functions L and f have not only the same function value, they also have the same slope at a.

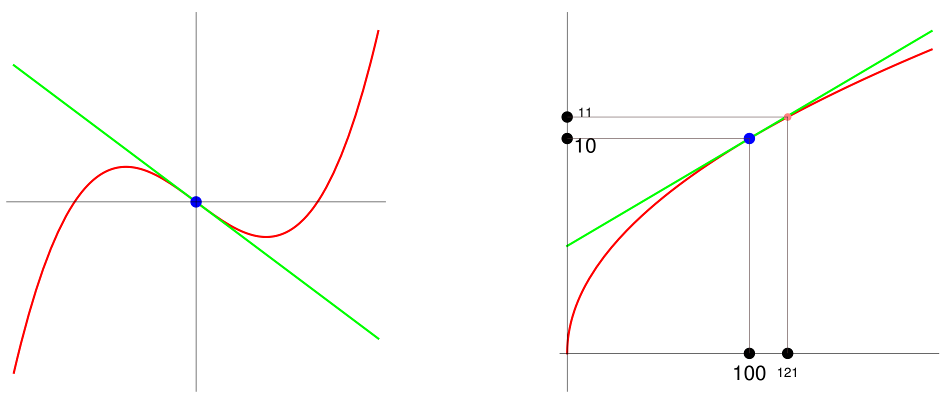


FIGURE 1. Left: The function $f(x) = x^3 - x$ and its linearization $y = f'(0)x + f(0) = -x$ at $a = 0$. Right: the function $f(x) = \sqrt{x}$ and its linearization $y = f'(100)x + f(100) = x/20 + 100$ at $a = 100$.

11.2. Lets look at $f(x) = x^2$ and $a = 10$. We have $f(10) = 100$ and $f'(x) = 2x$ which gives $f'(10) = 20$. The **linearization** = **linear approximation** of f at $a = 10$ is

$$L(x) = f(10) + f'(10)(x - 10) = 100 + 20(x - 10) = 20x - 100 \ .$$

Compare $f(x), L(x)$ near $a = 10$: we have $f(11) = 121$ and $L(11) = 20 \cdot 11 - 100 = 120$.

11.3. Why would we replace a fine function with something else? The reason is that the function $f(x)$ might be complicated. Lets take $f(x) = \sqrt{x}$ and try to compute $\sqrt{104}$ without a computer. We know that for $a = 100$, the square root can be computed as $f(a) = 10$. We also know that $f'(x) = 1/(2\sqrt{x})$ and so that $f'(a) = 1/(2 \cdot 10) = 1/20$. The linearization is now

$$L(x) = 10 + \frac{(x - 100)}{20}, L(104) = 10 + \frac{4}{20} = 10 + \frac{1}{5} = \frac{51}{5} = 10.2 \ .$$

The actual number is 10.19804. Not too shabby, mate!

Homework due Friday Feb 16, 2024

Problem 11.1: Use linear approximation to estimate $\sqrt{103}$ and $\sqrt{97}$. Your results need to be fractions.

Problem 11.2: a) Use linearization to find the 5th root of 34.
b) Impress your friends and compute the cube root of 1000001 to 10 digits in your head (of course using linearization!)

Problem 11.3: You start a business by making **edible iphones**. You estimate that the cost for x cases will be $C(x)$ dollars, where the cost function is

$$C(x) = 20\sqrt{x} + 100$$

Its derivative $C'(x)$ is called the **marginal cost function**.
a) Use linearization to estimate $C(17)$.
b) Economists restate the notion of marginal cost by saying that $C'(x)$ is the cost of producing one more item when producing x items. Explain why this is not exactly true but why this is a reasonable statement.

FIGURE 2. An edible iphone (AI generated).

Problem 11.4: Use linear approximation to estimate $e^{0.01}$. Is your estimate an overestimate or underestimate?

Problem 11.5: a) We all know that $\log'(x) = \ln'(x) = 1/x$. For $f(x) = \log_b(x)$ to the base b, we can use that $g(x) = e^{\log(b)x}$ has the derivative $g'(x) = \log(b)g(x)$. Argue using linearization to see that $\boxed{f'(x) = 1/(x\ln(b))}$.
b) Use linear approximation to estimate $\log_{10}(1000001)$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 12: Maxima and Minima

12.1. Pierre Fermat made a simple but profound observation: if $f'(x)$ is not zero, then x can not be a maximum nor be a minimum. His reasoning was: if you make a step h then you end up at $f(x+h) \sim L(x+h) = f(x) + hf'(x)$. Indeed, we all know that if there is a slope and do a step we end up a bit higher.

12.2. Lets call a point x a **local maximum** of f if $f(y) \leq f(x)$ for all y near enough to x. The function $f(x) = x^3 - 2x$ for example has a local minimum at $x = 1$ and a local maximum at $x = -1$. The observation of Fermat is equivalent to:

Fermat's principle: If a differentiable function f has a local maximum or minimum at x, then $f'(x) = 0$.

12.3. The function $f(x) = x^2$ for example has the derivative $f'(x) = 2x$. This is zero at $x = 0$, the minimum of f. Note that the converse of Fermat's statement is not necessarily true: if $f'(x) = 0$, then x does not need to be a maximum or minimum. The standard example is $f(x) = x^3$. We have $f'(x) = 3x^2$ which is zero at $x = 0$. But $x = 0$ is neither a maximum nor minimum of f.

12.4. A point x is called a **critical point** of f, if $f'(x) = 0$. Critical points are important because they are **candidates for maxima and minima**.

12.5. The next test allows to see whether we have a maximum or minimum. The derivative should exists near a but not necessarily at a. Like for $f(x) = |x|$ and $a = 0$.

First derivative test: If a is a critical point of f and the slope $f'(x)$ changes from negative to positive at a then a is a local minimum. If $f'(x)$ changes from positive to negative at a, then a is a local maximum. If $f'(x)$ does not change sign, the a is neither a local maximum nor local minimum.

12.6. Second derivatives help. It assumes that the second derivative exists at a.

Second derivative test: If a is a critical point of f and $f''(a) > 0$, then f is a local minimum. If $f''(a) < 0$, then f is a local maximum. If $f''(a) = 0$, the test is inconclusive.

Homework

This PSet is due Wednesday February 21, 2024.

Problem 12.1: Find the critical points of the following two functions
a) $f(x) = x^4 - 4x^3 + 4x^2$.
b) $f(x) = x^4(x-3)^2$.

Problem 12.2: Use the second derivative test to determine the nature of the critical points in the same two functions:
a) $f(x) = x^4 - 4x^3 + 4x^2$.
b) $f(x) = x^4(x-3)^2$.

Problem 12.3: What does the first derivative test tell you about the behavior at the critical points in the two cases
a) $f(x) = x^4 - 4x^3 + 4x^2$.
b) $f(x) = x^4(x-3)^2$.
In each case, is there a point, where the first derivative test gives more information than the second derivative test?

Problem 12.4: Find all the critical points and determine whether it is a local maximum, a local minimum or neither. You can use either test. You will see that some of the cases are a bit unusual.
a) $f(x) = e^2x - e^x$
b) $f(x) = e^x + x$
c) $f(t) = t^4 + t^3$
d) $f(t) = |2t - 8|$
e) $f(x) = 5$.

Problem 12.5: a) Both the first and second derivative test do not work for the **tamed devil function** $f(x) = x\sin(1/x)$ at $x = 0$. Why not? (Since we have no chain rule yet, you can certainly look up the first and second derivative using a tool like Wolfram alpha.).
b) Function $f(x) = \arcsin(\sin(x))$ has appeared in our ground hog movie. Where are the maxima and minima? To do so, plot the function $f(x)$ and its derivative $f'(x)$ and use one of the derivative tests at the maxima and minima.

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Unit 13: Global maxima

13.1. In applications, the domain of the function can be limited. For example, if we want to find the rectangle of width x and length y that maximizes the area xy given a circumference of $2x+2y=4$, we have to find the maximum of $f(x)=x(2-x)=2x-x^2$. But obviously, the width can not be negative, nor can it be larger than 2 without $y=2-x$ becoming negative. We need to maximize on the closed interval $[0,2]$.

13.2. A point $x \in [a,b]$ is a **global maximum** if there is no point y in $[a,b]$ for which $f(y) > f(x)$. The point x is a **global minimum** if x is a global maximum for $-f$. Here is a theorem of Bolzano: [1]

Extreme value theorem: A continuous function f on $[a,b]$ has a global maximum and a global minimum.

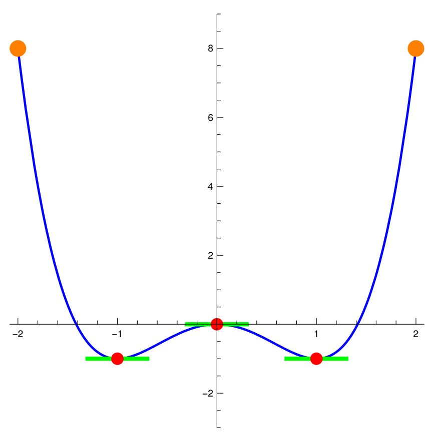


FIGURE 1. A prototype example $f(x)=x^4-2x^2$. There are two local minima $-1,1$ and 3 local maxima. The minima are also global. Only the boundary points are global maxima. Note that $f'(x)$ does not need to be zero at the boundary points. This example is iconic and a Goldstone boson picture used to explain **spontaneous symmetry breaking** in physics.

[1]We do not use the term "absolute maximum", as it suggests to look a maximum of $|f|$.

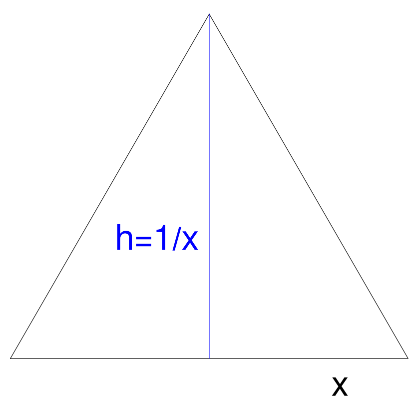
13.3. Problem: Find the global maximum and minimum of $f(x) = x^4 - 2x^2$ on $[-2, 2]$. **Solution**: $f'(x) = 4x^3 - 4x = 0$ for $x = -1, 0, 1$. The second derivative $f''(x) = 12x^2 - 4$ is positive at $x = -1, 1$ and negative at $x = 0$. We have two local minima $-1, 1$ and one maximum 0. Include the boundary points $-2, 2$. The total list of points to consider is $\{-2, -1, 0, 1, 2\}$. The function values are $\{8, -1, 0, -1, 8\}$. The points $\{-1, 1\}$ are global minima, the points $-2, 2$ are global maxima.

To find a global maximum of f on $[a, b]$, make a list of local maxima in the interior (a, b) using the first or second derivative test, then collect the boundary points as candidates. Among this combined list, chose where f is maximal.

13.4. Back to the rectangle problem with $f(x) = x(2-x) = 2x - x^2$ on $[0, 2]$. We find the local maxima using the second derivative test $f'(x) = 2 - 2x = 0$ for $x = 1$. The boundary point values are $f(0) = 0$ and $f(2) = 0$. The later are the global minima.

13.5. Here is a similar but a bit more complex problem:

Which isosceles triangle of height h and base $2x$ and area $xh = 1$ has minimal circumference $2x + 2\sqrt{x^2 + h^2}$?



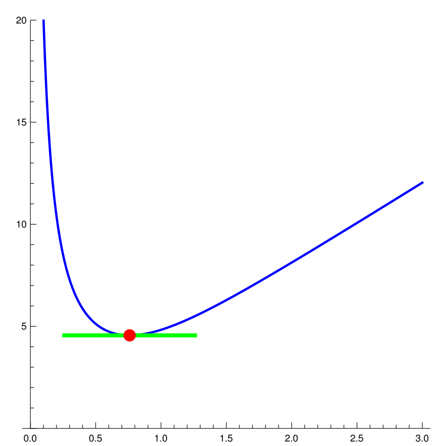


FIGURE 2. A function $f(x) = 2x + 2\sqrt{x^2 + 1/x^2}$ gives the circumference of a triangle with base $2x$ and height $h = 1/x$. We want to find the minimal circumference. f is defined on $(0, \infty)$. There is a unique minimum. There are no boundary points to consider. This example is a special case of the **isoperimetric inequality**: a **n-gon** of area 1 with minimal circumference must be a regular n-gon. Symmetry rules!

13.6. We have to extremize the function $f(x) = 2x + 2\sqrt{x^2 + 1/x^2}$. The base length can not be negative, nor zero so we have to look at the problem on $(0, \infty)$. There are no boundary points to consider so that only candidates for minima are places, where $f'(x) = 0$. The only positive solution of of $f'(x) = 0$ is $x = 1/3^{1/4}$. This means $h = 3^{1/4}$. One can check that $x^2 + h^2 = 4x^2$ so that this is an equilateral triangle.

13.7. About the proof of the extreme value theorem: pick a point x_1. If there is no point with a larger value, it is the global maximum. If not, there is a point x_2, where $f(x_2) > f(x_1)$. If there is no other point with larger value, then x_2 is the global maximum. The alternative is that there is an other point x_3 with $f(x_3) > f(x_2)$. Continuing like this we either end up at a global maximum or then find a sequence of numbers x_n with increasing f values. Now split $I_1 = [a, b]$ into two intervals of equal length $(b-a)/2$. The sequence x_n has to visit one (or both) of them infinitely often. Pick such an interval I_2 and renumber the x_k which hit that interval. Again split the interval into two intervals of length $(b-a)/4$ and pick one I_3 for which there are infinitely many points x_k. We have now a nested sequence of intervals of smaller and smaller length. The intersection of all these intervals is a single point $[x, x]$. This point is a global maximum. The same argument shows that if x_n would be unbounded, then the function would not be continuous at x.

Homework: Due Friday 2/23/2024

Problem 13.1: Find the global maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 7$ on $[-2, 6]$.

Problem 13.2: Find all the global maxima and minima of $f(x) = 3x^{2/3} - x$ on $[-1, 1]$.

Problem 13.3: Find all the global maxima and minima of $f(x) = t^{-1} + 2t^{-2}$ on $[1, \infty)$.

Problem 13.4: Find all the global maxima and minima of $|\ln(x+3)|$ on $[-3, \infty]$.

Problem 13.5: Does there exist an example of a continuous function on $[-2, 2]$ with the given properties or not? If yes, pick an example from the given list A-D.
A) $f(x) = x^3$
B) $f(x) = 3$
C) $f(x) = x^2$
D) $f(x) = -x^2$

Question	Yes	No	Example if yes A-D
a) f has a global max but no global min			
b) f has only global max and global min			
c) f has a two global max and one global min			
d) f has a one global max and one global min			

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Unit 14: Applications

14.1. Most laws in physics or chemistry are based on extremization: systems settle at minimal energy, systems maximize entropy, light minimizes action or soap bubbles minimize surface area. An other fundamental law of nature states that students optimize their day in order to have maximal time for practicing calculus.

14.2. A **soda can** is a cylinder of volume $\pi r^2 h$. Its **surface area** $2\pi rh + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is 2π, we can solve the equation for $h = (1-r^2)/r = 1/r - r$ **Solution:** The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.

14.3. Take a piece of paper 2×2 inches. If we cut out 4 squares of equal side length x at the corners, we can fold up the paper to a tray with width $(2-2x)$ length $(2-2x)$ and height x. For which $x \in [0,1]$ is the tray volume maximal?
Solution: The volume is $f(x) = (2-2x)(2-2x)x$. To find the maximum, we need to compare the critical points which is at $x = 1/3$ and the boundary points $x = 0$ and $x = 1$.

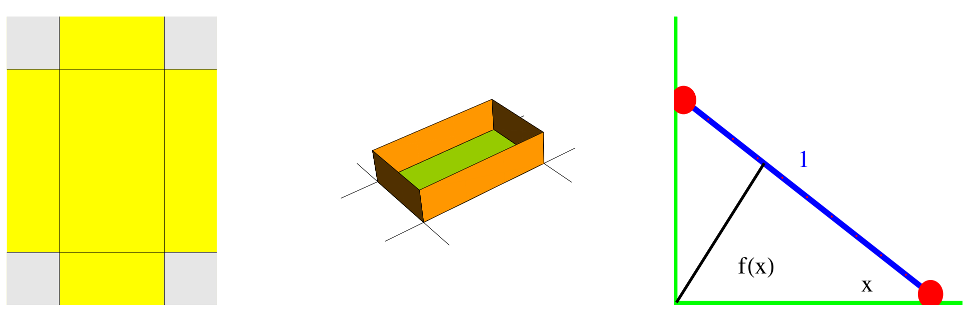


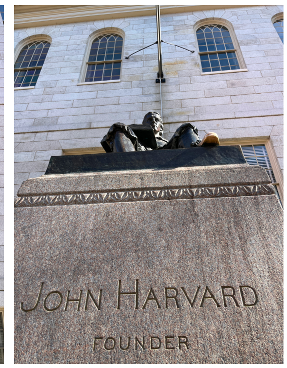
FIGURE 1. Finding the largest tray and the ladder problem.

14.4. Problem: A ladder of length 1 is one side at a wall and on one side at the floor. The distance of the ladder to the corner is $f(x) = \sin(x)\cos(x)$. [An other way to see this: the triangle has area $\sin(x)\cos(x)/2$. Since the hypotenuse is 1, the height must be $\sin(x)\cos(x)$.] Find the angle x for which $f(x)$ is maximal. **Solution** :

The distance is $f(x) = \sin(x)\cos(x) = \sin(2x)/2$ which has a local maximum when $f'(x) = 0$. The maximum is at $x = \pi/2$.

14.5. On February 21st, Oliver joined the tourists looking at the John Harvard Statue. For away or very close the viewing angle becomes small. There is an optimal distance, where the viewing angle is maximal. Find this distance. We work on this in a worksheet. Here are some pictures which show this. First from very far, then closer and closer and finally photographed from very close. Obviously there is an angle which is optimal.



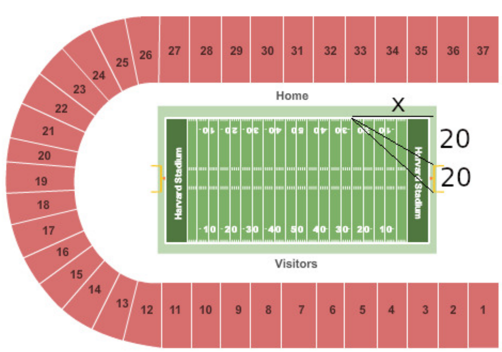


14.6. Problem We flip **coins**. The probability of hitting head is p. The entropy of this situation is defined as $S(p) = -p\log(p) - (1-p)\log(1-p)$. Which coin probability maximizes entropy? We will do this computation in class. Even more interesting is the minimization of **free energy** $F(p) = H - TS = ap + b(1-p) + Tp\log(p) + T(1-p)\log(1-p)$ which gives the **Gibbs distribution** $p = e^{b/T}/(e^{a/T} + e^{b/T})$.

Homework

This PSet is due Monday February 26, 2024.

Problem 14.1: Here is a problem very similar to the statue problem but a problem in football (as we have some football players in class): from which point on the sideline of the Harvard stadium does the goal post appear under the largest angle? We need to maximize $f(x) = \arctan(40/x) - \arctan(20/x)$.



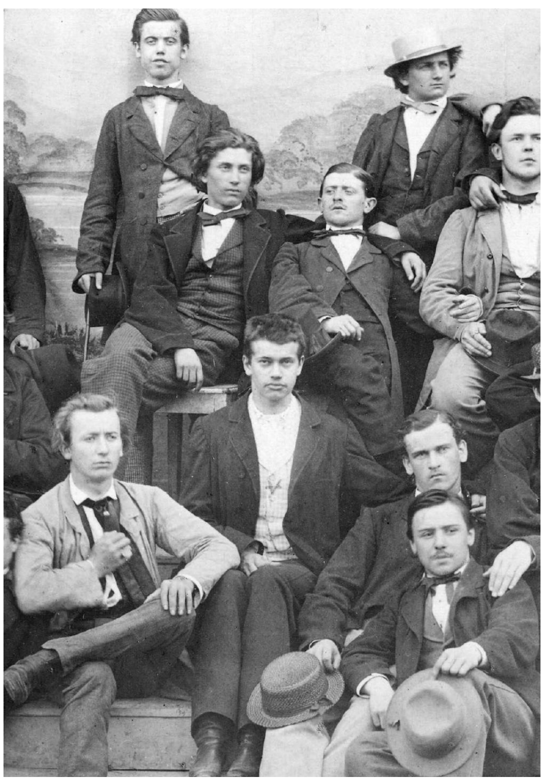
Problem 14.2: Which rectangle of dimensions x, y inscribed in $x^2/4 + y^2 = 1$ has maximal area $f = xy = 2x\sqrt{4 - x^2}$?

Problem 14.3: **Mathcandy.com** manufactures spherical candies of effectiveness $f(r) = A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r. We want to have the largest effectivity of $f(r) = 4\pi r^2 - 4\pi r^3/3$.

Problem 14.4: The function $S(x) = -x\ln(x)$ is called the **entropy** of the probability x. Find the probability $0 < x \leq 1$ which maximizes entropy.

Problem 14.5: Find the global minimum of the **Helmholtz free energy** $G = H - TS$, where $T = 10$ is temperature, $S(x)$ is the entropy function in problem 14.4 and $H = x$ is an **internal energy**.
P.S. One of the most important principles in science is that nature tries to maximize entropy or minimize free energy.

Entropy has been introduced by Ludwig Boltzmann. It is important in physics and chemistry. $\boxed{S = k\log(W)}$ which one can find on his tombstone, is interpreted using "Wahrscheinlichkeit" $W(p) = 1/p$. Take the expectation of $\log(W)$ to get $S = -k\sum_p p\log(p)$. Note the use of log and not ln. Claude Shannon (a local) introduced the same entropy function in information theory. The picture to the right shows Hermann von Helmholz (1812-1894).



Boltzmann (1844-1906)

Helmholtz (1812-1894).

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024

Unit 15: Review

Overview

A function f is **continuous** at a if there is $b = f(a)$ such that $\lim_{x\to a} f(x) = b$. It is continuous on the interval $[a, b]$ if it is continuous on every point in $[a, b]$. The enemy of continuity are **jumps**, **infinity** and oscillation. The **first derivative** f' tells whether the function is **increasing** or **decreasing**. It is defined as the limit $[f(x+h)-f(x)]/h$ as $h \to 0$. The **second derivative** tells whether the function is **concave up**, **concave down**. Roots of f' are critical points. Roots of f'' can lead to inflection points, points where the concavity changes. The graph of the line $L(x) = f(a)+f'(a)(x-a)$ is tangent to the graph of f at a. A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$. If $f' > 0$ then f is **increasing**, if $f' < 0$ it is **decreasing**. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a **horizontal tangent**. To determine whether a point is a maximum or minimum, use either the **first derivative test** (change of f' near x) or the **second derivative test** (look at the sign of $f''(x)$). To maximize or minimze f on an interval $[a, b]$, find all critical points inside the interval, evaluate f on the **boundary** $f(a), f(b)$ and then compare the values to find the global maximum. If f is not differentiable somewhere, also include these **singular points** as candidates (like for $|x|$). To compute limits for indeterminate forms $0/0$ or ∞/∞, use **Hospital's theorem**: $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(a)/g'(a)$. To **estimate** $f(x)$ near a use **linearization** $f(x) \sim f(a)+f'(a)(x-a)$. A continuous function on $[a, b]$ has both a **global max** and global min by the **extreme value theorem**. The **fundamental theorem of trigonometry** is $\lim_{x\to 0} \sin(x)/x = 1$. To perform differentiation, master product and quotient rule.

Algebra reminders

Healing:	$(a + b)(a - b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)$
Denominator:	$1/a + 1/b = (a + b)/(ab)$
Exponential:	$(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b\ln(a)}$
Logarithm:	$\ln(ab) = \ln(a) + \ln(b)$. $\ln(a^b) = b\ln(a)$
Trig functions:	$\cos^2(x) + \sin^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots:	$a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	$x^3 + 2x^2 + 3x + 1$	Exponential	$5e^{3x}$
Rational functions	$(x+1)/(x^3 + 2x + 1)$	Logarithm	$\ln(3x)$
Trig functions	$2\cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$
$f(x) = c$	0
$f(x) = x^n$	nx^{n-1}
$f(x) = e^{ax}$	ae^{ax}
$f(x) = \cos(ax)$	$-a\sin(ax)$
$f(x) = \arctan(x)$	$1/(1+x^2)$

$f(x)$	$f'(x)$
$f(x) = 1/x$	$-1/x^2$
$f(x) = \sin(ax)$	$a\cos(ax)$
$f(x) = \tan(x)$	$1/\cos^2(x)$
$f(x) = \ln(x)$	$1/x$
$f(x) = \sqrt{x}$	$1/(2\sqrt{x})$

Differentiation rules

Scaling rule	$(d/dx f(cx) = f'(cx)$.	Translation rule	$(d/dx f(x+a) = f'(x+a)$.
Addition rule	$(cf+g)' = cf' + g'$.	Quotient rule	$(f/g)' = (f'g - fg')/g^2$.
Product rule	$(fg)' = f'g + fg'$.	Easy rule	simplify before deriving

Limit examples

$\lim_{x\to 0} \sin(x)/x$	l'Hospital 0/0	$\lim_{x\to 1}(x^2-1)/(x-1)$	heal
$\lim_{x\to 0}(1-\cos(x))/x^2$	l'Hospital 0/0 twice	$\lim_{x\to\infty} \exp(x)/(1+\exp(x))$	l'Hospital
$\lim_{x\to 0}(1/x)/\ln(x)$	l'Hospital ∞/∞	$\lim_{x\to 0}(x+1)/(x+5)$	no work necessary

Is $1/\ln|x|$ continuous at $x = 0$? Answer: yes with $f(0) = 0$

Is $\ln(1/|x|)$ continuous at $x = 0$. Answer: no.

$\lim_{x\to 1}(x^{1/3} - 1)/(x^{1/4} - 1)$. Answer: 4/3.

$\lim_{x\to 0} \frac{(e^x-1)(\sin(5x))}{e^{3x}-1} \sin(7x)$. Answer: 35/3.

$\lim_{x\to 0} \frac{x^{10000}-1}{x^{2000}-1}$. Answer: $10000/20000 = 5$.

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INTRODUCTION TO CALCULUS

MATH 1A

Unit 16: Mind hacks

16.1. Before we move on to a new topic, it is good to reflect a bit on the work we have done so far and also learn from the first midterm. Doing mathematics is not only about "adding content", but a "gym for your mind" in general. Even more important than content or "algorithmic knowledge" is "meta knowledge", knowing how to think and knowing how to work."

[1]

16.2. In computer science lingo, we mean with the later that we want to "tweak our operating system". The analog of "knowledge" are the files in your computer. The analog of "doing computations" is "running programs". But a thousand times more important than these two things is how your knowledge is organized, accessed, found and linked.

16.3. Lets look at Unix, a powerful, simple and efficient operating system. Unix knows the paradigms simplicity, generality and clarity. It uses small units of procedures which can be combined to make larger things happen. The command "find" for example allows to find everything fast, "grep" allows to filter things quickly. The simple building blocks can then like lego pieces put together without having to build new programs for each combination. More importantly, programs can write programs. The program TeX for example which is used for processing this very text, has been written by an

[1] illustration AI generated

other language WEB. Donald Knuth wrote a WEB program which produced pascal code for TeX.

A top 10 list

16.4. I regularly reflect improving "thinking". This usually goes under the name "work habits".

Principle: 1) Learn from mistakes
2) Improve work conditions
3) Turn tricks into methods
4) Use background processes
5) Keep important things in memory
6) List concepts that are not understood
7) Talk it out, even with AI
8) Keep mind maps
9) Focus on the essentials
10) Learn to absorb setbacks

How to solve

16.5. The mother of all problem solving books is Polya's "How to solve it" which was published in 1945. If you read and absorb this book, you immediately get measurably stronger in math. Still after more than 70 years, it is the best. Here are the now famous **Polya principles**:

Polya principles
1. **Understand** the problem: unknowns, data, draw figure.
2. Devise a **plan**: similar or related problem?
3. **Carry out** the plan: check each step.
4. **Examine** the solution: can other problems be solved as such?

16.6. Here is a problem taken from the book of Polya. We will learn how to solve this next time. It is related to the bottle calibration problem we have seen earlier in the class.

Problem Problem:: Water is flowing with a constant rate of one cubic meter per second into a conical vessel $x^2 + y^2 = z^2, z \geq 0$. At which rate is the water level rising if the water depth is z meters?

16.7. Lets try in class to work on this problem. It will lead us to related rates covered next time.

16.8.

Tao's deformation principles
a. Consider special, extreme or degenerate cases.
b. Solve a simplified version of the problem
c. Formulate a conjecture
d. Derive intermediate steps which would get it.
e. Reformulate, especially try contraposition.
f. Examine solutions of similar problems
g. Generalize the problem

Homework

This PSet is due Monday March 4, 2024:

Problem 1: One of the continuity questions was whether $x \ln |x|$ is continuous everywhere. Most of the class answered that the limit $x = 0$ is not defined.
a) Use l'Hospital to show that $\lim_{x \to 0} x \ln |x| = 0$. This is of the form $0 \cdot \infty$. You need to write it in as an indefinite form ∞/∞ first.
b) Simplify $e^{x \ln |x|}$ so that it has not more than 3 symbols. We have looked at this expression before.

Problem 2: The table where one has to identify some features about functions was solved poorly. Please solve that problem again. Reflect for each topic "even, odd, periodic or invertible" why you were stuck.

Problem 3: Look over your first midterm and identify **one general point** which needs improvement. What is the most important concept or or more general pattern you need to work on for next time?

Problem 4: Solve the Polya problem mentioned in this text: "Water is flowing with a constant rate of one cubic meter per second into a conical vessel $x^2 + y^2 = z^2, z \geq 0$. At which rate is the water level rising if the water depth is z meters?" This is preparation for next week, when we will look at related rates problems.

Problem 5: Choose one of the **mind performance hacks** distributed in class (a selection from the book) Pick the one which does most appeal to you and summarize in a short paragraph.

[2]

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[2]For more literature, see https://people.math.harvard.edu/knill/creativity/books.html

Unit 17: Chain Rule

17.1. For the derivative of the composition of functions like $f(x) = \sin(x^7)$, we can not use the product rule. The functions don't hold hands like in a product, they are "chained" in the sense that we evaluate first x^7 then apply the sin function to it. In order to differentiate, take the derivative of the x^7 then multiply this with the derivative of the function sin evaluated at x^7. The answer is $\cos(x^7)7x^6$. Here is the rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \; .$$

17.2. The chain rule follows from the following identity that is true for **all** functions f, g such that $h, H = g(x+h) - g(x) > 0$. (If $g(x+h) = g(x)$, we just would get zero on the left.)

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x)+H) - f(g(x))]}{H} \cdot \frac{H}{h} \; .$$

If f, g are differentiable we can take the limit $h \to 0$, which gives $H \to 0$. The first part goes to $f'(g(x))$ and the second factor goes to $g'(x)$.

17.3. Let us look at some examples.

Problem: Find the derivative of $f(x) = (4x^2 - 1)^{17}$.
Solution The inner function is $g(x) = 4x^2 - 1$ with derivative $8x$. We get $f'(x) = 17(4x - 1)^6 \cdot 8x$.
Remark. Expansion of $(4x^2 - 1)^{17}$ would have avoided the chain rule. But we would have been using a **pain rule** .

Problem: Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution**: applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

For linear functions $f(x) = ax+b, g(x) = cx+d$, the chain rule can readily be checked: we have $f(g(x)) = a(cx+d)+b = acx+ad+b$ which has the derivative ac. This agrees with the definition of f times the derivative of g. Using this and linearization can prove the chain rule too.

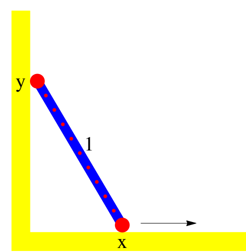
$f(x) = \sin(x^2 + 3)$. Then $f'(x) = \cos(x^2 + 3)2x$.

$f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)$.

The name "chain rule" is because we can chain even more functions together:

Problem: Let us compute the derivative of $\sin(\sqrt{x^5 - 1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5 - 1}$ gives $\cos(\sqrt{x^5 - 1})\frac{d}{dx}\sqrt{x^5 - 1}$. Apply now the chain rule again for the derivative on the right hand side.

Example: remember the **falling ladder problem**, where a ladder of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0, y)$ on the wall with $(x, 0)$ on the floor. We want to express y as a function of x. We have $y = f(x) = \sqrt{1 - x^2}$. Taking the derivative, assuming $x' = 1$ we get $f'(x) = -2x/\sqrt{1 - x^2}$. Infinite speed at the end. The ladder will definitely break the sound barrier.



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

Lets push that to the extreme and differentiate

$$f_{11}(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x))))))))))) .$$

Here is the poetic formula obtained when running this in Mathematica:

$$\mathrm{D}[\mathrm{Last}[\mathrm{NestList}[\mathrm{Exp}, \mathrm{x}, 11]], \mathrm{x}]$$

$$\exp\left(e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{x}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{x}}}}}}} + e^{e^{e^{e^{e^{e^{x}}}}}} + e^{e^{e^{e^{e^{x}}}}} + e^{e^{e^{e^{x}}}} + e^{e^{e^{x}}} + e^{e^{x}} + e^{x} + x\right)$$

Note the exponential at the left. This can also be written as a product. In class we broke the record and wrote down the derivative of a chain of 12 exponentials and because $f_n' = f_n f_{n-1}'$ we have $f_{12}' = f_{12}f_{11}f_{10}f_9f_8f_7f_6f_5f_4f_3f_2f_1$.

Problem: Find the derivative of $1/\sin(x)$ using the quotient rule as well as using the chain rule using the function $g(x) = 1/x$. **Solution** $-\cos(x) \cdot 1/\sin^2(x)$.

Homework: Due Wed, 3/6/2024

Problem 17.1: Find the derivatives of the following functions:
a) $f(x) = xe^{-x^2}$
b) $f(x) = \ln(\ln(x))$
c) $f(x) = \cot(x^{17})$
d) $f(x) = x/(4 + x^2)$
e) $(\sin(x) + \cos(x))^{-3}$

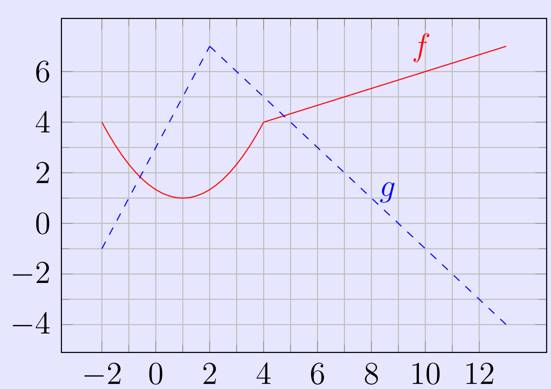
Problem 17.2: Find the derivatives of the following functions at $x = 1$.
a) $\sqrt{x^2 - 1}$
b) x^x.
c) $(1 + x^3)^{100}$
d) $\sin(\sin(\sin(x)))$

e) $\sin(\sin(\sin(\sin(\sin(x)))))$.

Problem 17.3: We also need to practice taking derivatives with respect to variables that are different from x.
a) Find the derivative of $e^{-t^2}\sin(t^4)$ with respect to t.
b) Find the derivative of $\cos(\sin(\cos(u^2)))$ with respect to u.

Problem 17.4: Lets go back to some rule we have already seen earlier.
a) Verify that
$$\frac{d}{dx}f(cx + b) = cf'(cx + b)$$
b) $\frac{d}{dx}[\ln(6x + 3) + \sin(3x + 5) + \cos(11x + 3)]$.

Problem 17.5: The following graph shows functions f and g. If $F(x) = f(g(x))$, find $F'(1)$ and $F'(8)$.



Unit 18: Implicit Differentiation

18.1. Assume we have a relation between x and y like

$$x^4y + xy^4 = 2x$$

and we also know that $x = 1$ and $y = 1$. Can we use this to get the derivative y' without actually solving for y?

18.2. The answer is yes. Just differentiate and use the chain rule:

$$4x^3y + x^4y' + y^4 + 4xy^3y' = 2$$

Now solve for y' to get $y' = [2 - 4x^3y - y^4]/[x^4 + 4xy^3]$. At $x = 1, y = 1$ we see the answer $-4/5$. This is really cool because we would not have been able to solve the above equation for y and differentiate that expression.

18.3. Lets look at the example $x^2 + 3y^2 = 4$. Can you find the derivative y' at $x = 1$ knowing $y = 1$? Solution. We have $2x+6yy' = 0$, so that $y' = -2x/6y = -2/6 = -1/3$. In this case we would have been able to solve for y and differentiate.

18.4. A cool application of the chain rule is to find the derivatives of inverses: **Example:** What is $\log'(x)$? Lets pretend we do not know this already but that we know the derivative of e^x as well as that log is the inverse of $e^x = \exp(x)$. [1] **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x))\log'(x) = x\log'(x)$. Setting this equal gives $x\log'(x) = 1$. Therefore $\log'(x) = 1/x$.

$$\frac{d}{dx}\log(x) = 1/x.$$

Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0, \pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$ and with $\arctan(x)$ the inverse of $\tan(x)$. The arctan is defined everywhere.

[1] $\log(x)$ stands also for $\ln(x)$ "logarithmus naturalis". Similarly as $\exp(x) = e^x$ it abbreviates. Almost all computer languages (Python, C, Perl, R, Matlab, Mathematica) use "log" not "ln". Paul Halmos called "ln" a childish notation which no mathematician ever used. I fought against ln like Don Quixote for 20 years and gave up. Just assume $\ln = \log$ like $e^x = \exp(x)$.

18.5. Example: Find the derivative of $\arcsin(x)$. **Solution**.
1. Step) Start with $\sin(\arcsin(x)) = x$.
2. Step) Differentiate both sides using the chain rule $\cos(\arcsin(x))\arcsin'(x) = 1$.
3. Step) Isolate $\arcsin'(x)$: $\arcsin'(x) = 1/\cos(\arcsin(x))$.
4. Step) Simplify: $1/\cos(\arcsin(x)) = 1/\sqrt{1-\sin^2(\arcsin(x))} = 1/\sqrt{1-x^2}$.

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \arctan'(x) = \frac{1}{1+x^2} .$$

Homework: Due 3/8/2024

Problem 18.1: You know $y^3 + x^2 y + x y^2 = 14$. Find y' at $x = 1$ knowing $y = 2$.

Problem 18.2: a) Find the derivative of $f(x) = 1/x$ by differentiating $x f(x) = 1$.
b) Compute $\text{arccot}'(x)$.

Problem 18.3: a) Find the derivative of $f(x) = \sqrt{x}$ by differentiating $f(x)^2 = x$.
b) Find the derivative of $f(x) = x^{m/n}$ by differentiating $f(x)^n = x^m$.

Problem 18.4: a) What is the derivative of $\arcsin(\arccos(x))$?
b) What is the derivative of $\arctan(\arctan(x))$?
c) Compute the derivative of $\arctan(\arctan(\arctan(x)))$.
P.S. When drawing out the graphs of these iterations we see a limiting function.

Problem 18.5: a) Compute $\text{arccosh}'(x)$. b) Compute $\text{arcsinh}'(x)$.

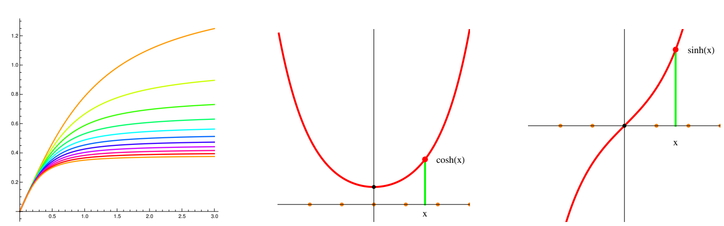


FIGURE 1. To the left, plots of $\arctan(x)$, $\arctan(\arctan(x))$, $\dots \arctan(\arctan(\dots\arctan(x)))$. In the middle, $\cosh(x) = (e^x + e^{-x})/2$, then $\sinh(x) = (e^x - e^{-x})/2$. We have $\cosh^2(x) - \sinh^2(x) = 1$.

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INTRODUCTION TO CALCULUS

MATH 1A

Unit 19: Related rates

19.1. Related rates problems solve relations between variables. Implicit differentiation is a special case. By taking derivatives of an equation, we get relations between variables. In all these problems, we have an $\boxed{\textbf{equation}}$ and a $\boxed{\textbf{rate}}$. You can then solve for the rate which is asked for.

19.2. Hydrophilic **water gel spheres** have volume $\boxed{V(r(t)) = 4\pi r(t)^3/3}$ and expand at a rate $\boxed{V' = 30}$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.

19.3. A **wine glass** has a shape $y = x^2$ and volume $\boxed{V(y) = y^2\pi/2}$. Assume we slurp the wine with constant rate $\boxed{V' = -0.1}$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi yy'(t)$ so that $y'(t) = -1/(\pi y)$.

19.4. A **ladder** has length 1. Assume slips on the ground away with constant speed $\boxed{x' = 2}$. What is the speed of the top part of the ladder sliding down the wall at the time when $x = y$ if $\boxed{x^2(t) + y^2(t) = 1}$. Differentiation gives $2x(t)x'(t)+2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

19.5. A **kid slides** down a slide of the shape $\boxed{y = 2/x}$. Assume $\boxed{y'(t) = -7}$. What is $x'(t)$? Evaluate it at $x = 1$. **Solution:** differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.

19.6. A **canister of oil** releases oil so that the area grows at a constant rate $\boxed{A' = 5}$. With what rate does the radius increase? **Solution.** See work sheet.

19.7. There is a saying: "everybody hates, related rates!". The reason is simple. If you look at related rates problems in textbooks, they are often hard to parse.

Related rates problems link quantities by a $\boxed{\text{rule}}$. These quantities can depend on time. To solve a related rates problem, differentiate the $\boxed{\text{rule}}$ with respect to time use the given $\boxed{\text{rate of change}}$ and solve for the unknown rate of change. To clarify, we have in this handout boxed the $\boxed{\text{rule}}$ and the known $\boxed{\text{rate of change}}$.

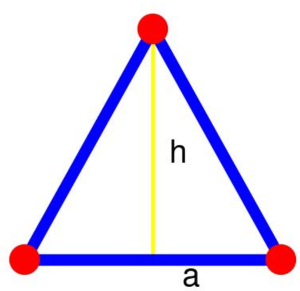
Homework: Due 3/18/2024

Problem 19.1: The **ideal gas law** $\boxed{pV = T}$ relates pressure p and volume V and temperature T. Assume the temperature $T = 50$ is fixed and $\boxed{V' = -3}$. Find the rate p' with which the pressure increases if $V = 10$.

Problem 19.2: Assume the **total production rate** P of a new tablet computer product for kids is constant $P = 100$ and given by the **Cobb-Douglas formula** $\boxed{P = L^{1/3}K^{2/3}}$. Assume labor is increased at a rate $\boxed{L' = 2}$. What is the cost change K'? Evaluate this at $K = 125$ and $L = 64$.

Problem 19.3: You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\boxed{\phi' = 5\pi/180 = \pi/36}$ radiants per second (which means 5 degrees per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\boxed{\tan(\phi) = x/h}$ to get ϕ for $x = 0$.

Problem 19.4: An **isosceles triangle** with base $2a$ and height h has fixed area $\boxed{A = ah = 1}$. Assume the height is decreased by a rate $\boxed{h' = -2}$. With what rate does a increase if $h = 1/2$?



Problem 19.5: There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $\boxed{V = \pi^2 r^4/2}$ increases at a rate $\boxed{V' = 100\pi^2 r^2}$. What is r'? Evaluate it for $r = 47$ (billion light years).

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Unit 20: Intermediate value theorem

20.1. Finding solutions to an equation $g(x) = h(x)$ is equivalent to find roots of $f(x) = g(x) - h(x)$.

Definition: If $f(a) = 0$, then a is called a **root** of f. For $f(x) = \sin(x)$ for example, there are roots at $x = 0, x = \pi$.

20.2. The function $f(x) = \log|x| = \ln|x|$ has roots $x = 1$ and $x = -1$. The function $f(x) = e^x$ has no roots.

Intermediate value theorem of Bolzano. If f is continuous on the interval $[a, b]$ and $f(a), f(b)$ have different signs, then there is a root of f in (a, b).

20.3. The proof is constructive and important: we can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at $c = (a + b)/2$. If $f(c) < 0$, then take $[c, b]$ as the new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of length $2^{-n}(b - a)$. Continuity assures that $f(u_n) - f(v_n) \to 0$ and that $f(u_n), f(v_n)$ have different signs. Both point sequences u_n, v_n converge to a root of f.

Verify that the function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root. **Solution.** The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. Use the theorem.

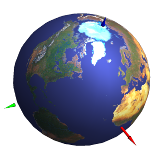
There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1 < 10$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem to the function $f(x) = x^x - 10$.

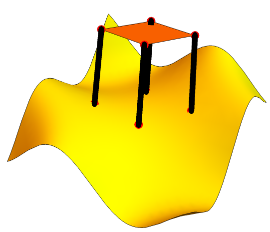
20.4. Earth Theorem. There is a point on the earth, where both temperature and pressure agree with the temperature and pressure on the antipode.

Proof. Draw an arbitrary meridian through the poles and let $f(x)$ be the temperature on that circle, where x is the polar angle. Define the function $g(x) = f(x) - f(x + \pi)$. If g is zero on the north pole, we have found our point. If not, $g(x)$ has different signs on the north and south pole. By the intermediate value theorem, there exists therefore an x, where $g(x) = 0$ and so $f(x) = f(x+\pi)$. For every meridian there is a latitude value $l(y)$ for which the temperature works. Define now $h(y) = l(y) - l(y+\pi)$. This function is continuous. Start with the meridian 0. If $h(0) = 0$ we have found our point. If not, then $h(0)$ and $h(\pi)$ have different signs. By the intermediate value theorem again, h has a root. There, both temperature and pressure agree with the antipode value.

20.5. Wobbly Table Theorem. On an arbitrary floor, a square table can be turned so that it does not wobble any more.

20.6. Proof. The 4 legs ABCD are located on a square in a plane. Let x be the angle of the line AC with with a coordinate axes if we look from above. Given x, the table can be positioned **uniquely**: the center of ABCD is on the z-axes, the legs ABC are on the floor and AC points in the direction x. Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. ($f(0) < 0$ is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h. This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.





20.7. The following is an application of the intermediate value theorem and also provides a constructive proof of the **Bolzano extremal value theorem** which we will see later.

Fermat's maximum theorem If f is continuous and has $f(a) = f(b) = f(a+h)$, then f has either a local maximum or local minimum inside the open interval (a, b).

20.8. The argument is to split the interval $[a, b]$ into two $[a, c]$ and $[c, b]$ of the same length. Now, $f(c) - f(a)$ and $f(b) - f(c)$ have different sign so that $g(x) = f(x + h/2) - f(x)$ has different signs $g(a)$ and $g(c)$ at the end points. By the intermediate value theorem there is a root of g in $[a, c]$ and therefore a point x in $[a, c]$ where $f(x) = f(x+h/2)$. This gives a new interval $[a_1, b_1]$ of half the size where the situation $f(a_1) = f(b_1)$ holds. Continuing like this we get a nested sequence of intervals $[a_n, b_n]$ which have size $2^{-n}h$. The limiting point is a maximum or minimum of f.

Homework: Due 3/20/2024

Problem 20.1: Find the roots for $-72 - 54x + 35x^2 + 15x^3 - 3x^4 - x^5$. You are told that all roots are integers.

Problem 20.2: Use the intermediate value theorem to verify that $f(x) = x^7 - 6x^6 + 8$ has at least two roots on $[-2, 2]$.

Problem 20.3: The "Queen's gambit" features two fine actors Anya Taylor-Joy and Thomas Brodie-Sangster (both sharing expressive wide eyes). Anya's height is 170 cm, Thomas height is 178 cm. Anya was born April 16, 1996, Thomas was born on May 16, 1990. Anya's and Thomas net worth are both estimated to be 3 Million.
a) Can you argue that there was a moment when Anya's height is exactly half of Thomas height?
b) Can you argue that there was a moment when Anya's age was exactly half the age of Thomas?
c) Can you argue that there as a moment when Anya's fortune was exactly half of Thomas fortune?
Argue with the intermediate value theorem or note a scenario where the statement is false.

Problem 20.4: Argue why there is a solution to
a) $5 - \sin(x) = x$, b) $\exp(7x) = x$, c) $\sin(x) = x^4$.
d) Why does the following argument not work:
The function $f(x) = 1/\cos(x)$ satisfies $f(0) = 1$ and $f(\pi) = -1$. There exists therefore a point x where $f(x) = 0$.
e) Does the function $f(x) = x + \log|\log|x||$ have a root somewhere? Argue with the intermediate value theorem.

Problem 20.5: a) Let $h = 1/2$. Find a h-critical point for the function $f(x) = |x|$. As defined in the text we look for a point for which $[f(x+h) - f(x)]/h = 0$.
b) Verify that for any $h > 0$, the function $f(x) = x^3$ has no h-critical point. There is no x, where $[f(x+h) - f(x)]/h = 0$ is possible.

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Unit 21: Finding Roots

21.1. Finding roots using the intermediate value theorem did not involve any differentiability. It worked for continuous functions. If a function is differentiable, we have more options. We can use linearization to find roots. We can also find intermediate points where the derivative is the average rate of change. The intermediate value will come handy when looking at the fundamental theorem.

21.2. Last time, we have seen to find roots of functions using a "divide and conquer" technique: start with an interval $[a, b]$ for which $f(a) < 0$ and $f(b) > 0$. If $f((a+b)/2)$ is positive, then use the interval $[a, (a+b)/2]$ otherwise $[(a+b)/2, b]$. After n steps, we are $(b-a)/2^n$ close to the root. If the function f is differentiable, we can do better and use the value of the derivative to get closer to a point $y = T(x)$. Lets find this point y. If we draw a tangent at $(x, f(x))$ and intersect it with the x-axes, then

$$f'(x) = \frac{f(x) - 0}{x - T(x)} .$$

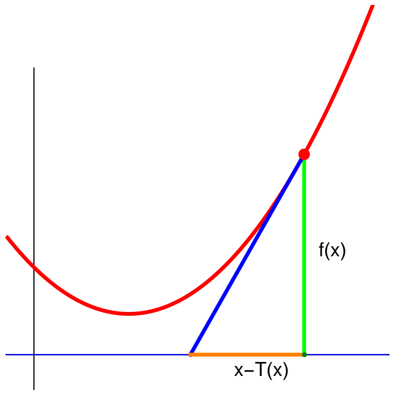
Now, $f'(x)$ is the slope of the tangent and the right hand side is "rise" over "run". If we solve for $T(x)$, we get

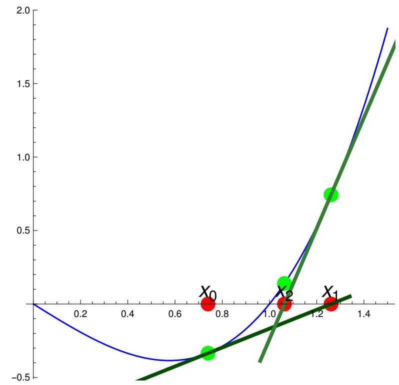
Definition: The **Newton map** of a function f is defined as

$$T(x) = x - \frac{f(x)}{f'(x)} .$$

21.3. The **Newton's method** applies this map a couple of times until we are sufficiently close to the root: start with a point x, then compute a new point $x_1 = T(x)$, then $x_2 = T(x_1)$ etc.

If p is a root of f such that $f'(p) \neq 0$, and x_0 is close enough to p, then $x_1 = T(x), x_2 = T^2(x)$ converges to the root p.





If $f(x) = ax + b$, we reach the root in one step.

If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0.

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$. In general, the number of correct digits double in each step.

In 4 steps we expect to have $2^4 = 16$ digits correct. Having a fast method to compute roots is useful. For example, in computer graphics, where things can not be fast enough. We will explore a bit in the lecture how fast the method is.

Lets compute $\sqrt{2}$ to 12 digits accuracy. We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned}
T(1) &= 1 - (1-2)/2 = 3/2 \\
T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\
T(17/12) &= 577/408 \\
T(577/408) &= 665857/470832\ .
\end{aligned}$$

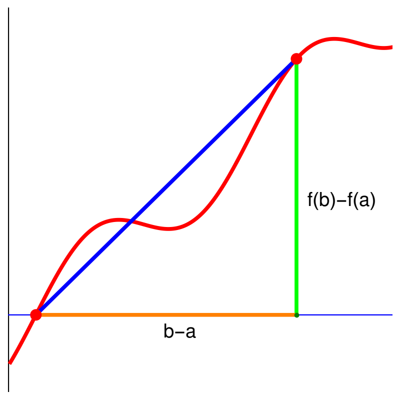
This is already $1.6 \cdot 10^{-12}$ close to the real root! 12 digits, by hand!

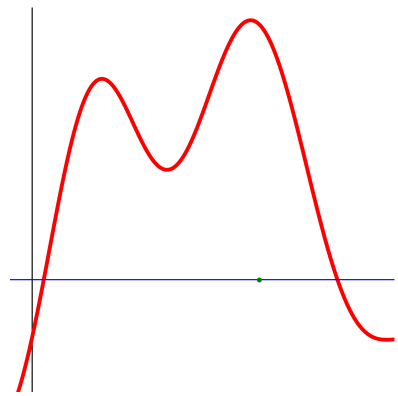
Mean value theorem

21.4. Unlike the intermediate value theorem which applied for continuous functions, the **mean value theorem** involves derivatives. Also here, we assume that f is differentiable unless specified. The mean value theorem is a consequence of the intermediate value theorem. It tells that the average rate of change is matched by an instantaneous rate of change somewhere.

Mean value theorem: Assume f is differentiable on $[a, b]$. Then there is a point x such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} \,.$$





Here are a few examples which illustrate the theorem:

Verify with the mean value theorem that the function $f(x) = x^2 + 4\sin(\pi x) + 5$ has a point where the derivative is 1.
Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1-0) = 5$.

A biker drives with velocity $f'(t)$ at position $f(b)$ at time b and at position a at time a. The value $f(b) - f(a)$ is the distance traveled. The fraction $[f(b) - f(a)]/(b - a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

21.5. Proof of the theorem: the function $h(x) = f(a) + cx$, where $c = (f(b) - f(a)/(b - a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are done. By tilting the picture, we have reduced it to a statement seen before:

Rolle's theorem: If $f(a) = f(b)$ then f has a critical point in (a, b).

Proof: If it were not true, then either $f'(x) > 0$ everywhere implying $f(b) > f(a)$ or $f'(x) < 0$ implying $f(b) < f(a)$.

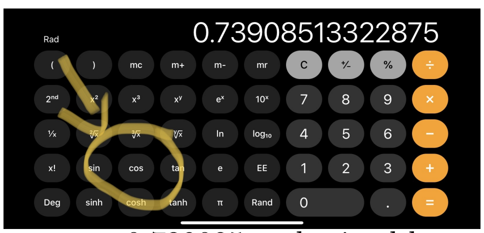
Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$.
Solution: The function is differentiable and non-negative. It is zero at $0, \pi$. By Rolle's theorem, there is a critical point.

Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g. [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is 1/2 where $g(1/2) = 21/2 > 0$.]

Homework, due 3/22/2024

Problem 21.1: Get the Newton map $T(x) = x - f(x)/f'(x)$ for:
a) $f(x) = (x-2)^2$
b) $f(x) = e^{5x}$
c) $f(x) = 2e^{-x^2}$
d) $f(x) = \cot(x)$.

Problem 21.2: The function $f(x) = \cos(x) - x$ has a root between 0 and 1. Starting with $x = 1$, perform the first Newton step.



Compare with the root $x = 0.739085...$ obtained by punching "cos" again and again

Problem 21.3: We want to find the square root of 102. We have to solve $\sqrt{102} = x$ or $f(x) = x^2 - 102 = 0$. Perform a Newton step starting at $x = 10$.

Problem 21.4: Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$. What happens if you apply the Newton steps starting with $x = 1$? Does the method converge?

Problem 21.5: We look at the function $f(x) = x^{10} + x^4 - 20x$ on the positive real line. Verify that the **mean value theorem** on $(1, 2)$ assures the there exists x, where $g(x) = f'(x) - [f(2) - f(1)] = f'(x) - 1018$. Now use a single Newton step starting with 1.5 to find an approximate solution to $g(x) = 0$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 22: Stability

22.1. In this lecture, we are interested how minima and maxima change when a parameter is changed. Nature, economies, or processes likes extrema. It turns out that if we change parameters, the outcome changes often in a non-smooth way. An economic parameter can change quickly for example. One calls this a catastrophe. This can be explained with mathematics. A key are **stable equilibria**, local minimum. Here is a general principle:

> If a local minimum disappears when we change an external parameter, the system settles in a new stable equilibrium. The new equilibrium can be far away from the original one.

22.2. To see this, let us look at the following optimization problem

22.3. Find all the minima and maxima of the function

$$f(x) = x^4 - x^2$$

Solution: $f'(x) = 4x^3 - 2x$ is zero for $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$. The second derivative is $12x^2 - 2$. It is negative for $x = 0$ and positive at the other two points. We have two local minima and one local maximum.

22.4. Now find all the extrema of the function

$$f(x) = x^4 - x^2 - 2x$$

There is only one critical point. It is $x = 1$. Lets introduce $f_c(x) = x^4 - x^2 - cx$. The first function was $f_0(x)$, the second function was $f_2(x)$.

22.5. When the first graph is morphed into the second example, the local minimum to the left has disappeared. Assume the function f measures the prosperity of some kind and c is a **parameter**. We look at the position of the first critical point of the function. Catastrophe theorists look at the following **assumption**:

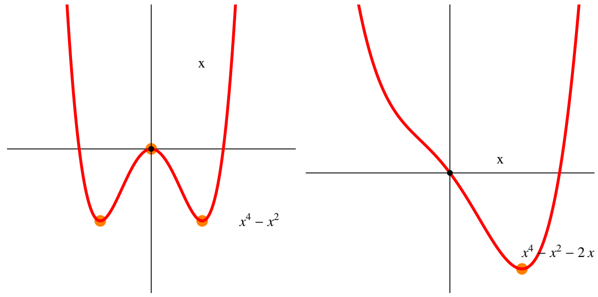
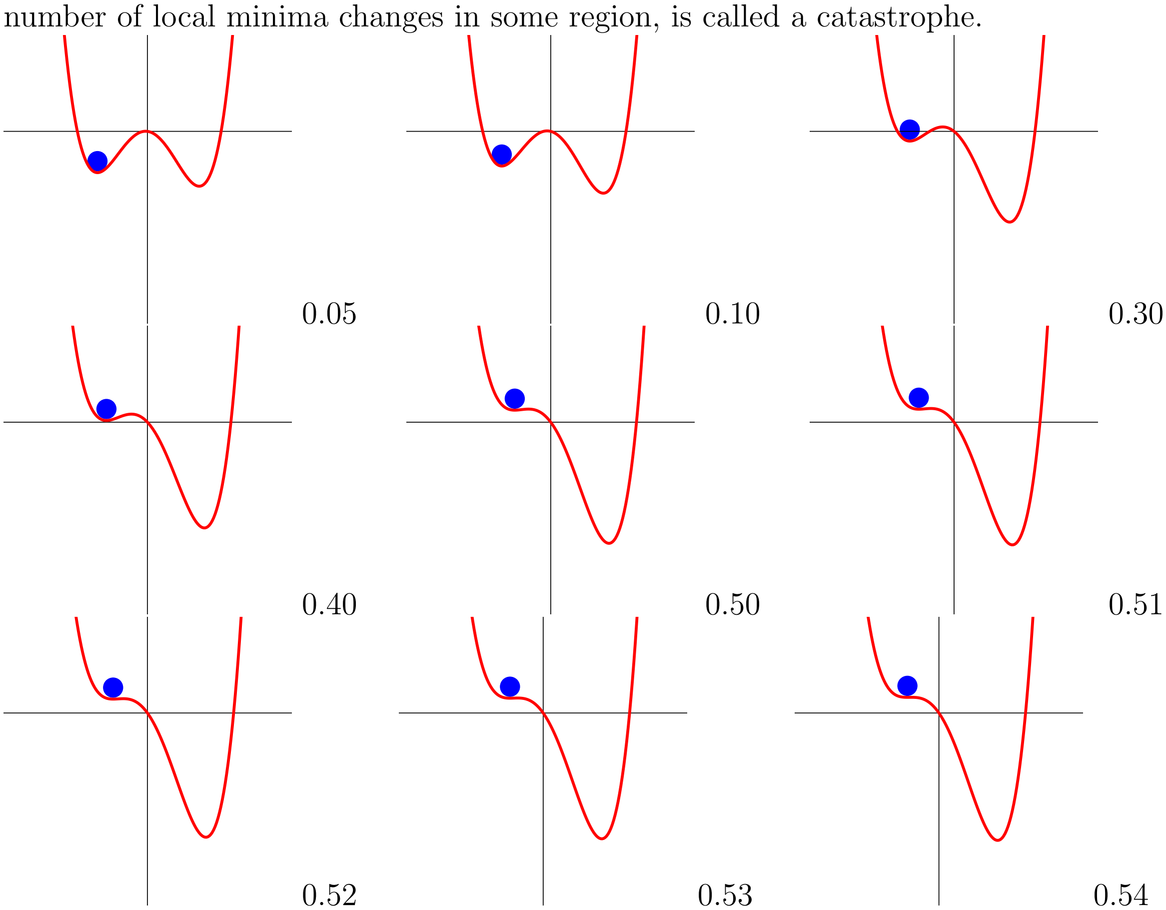


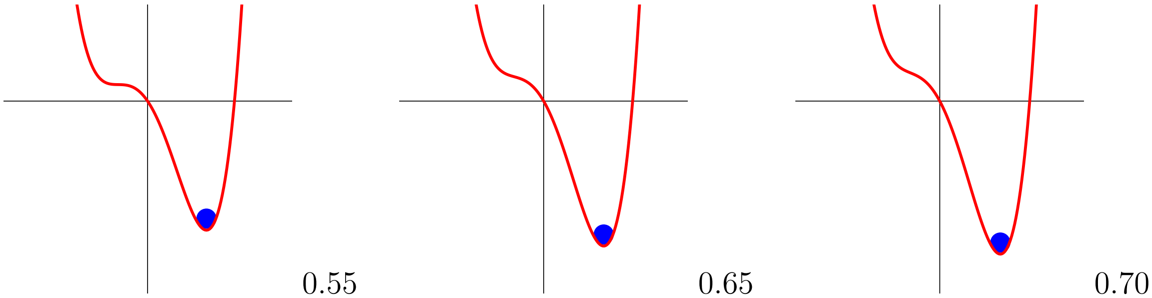
FIGURE 1. The function $f_0(x) = x^4 - x^2$ and $f_2(x) = x^4 - x^2 - 2x$.

22.6. Assume the function $f_c(x)$ depends on a parameter c the minimum, **stable equilibrium** depends on this parameter c. stable minimum disappears the system settles in general in an other stable equilibrium.

Definition: A parameter value c_0 at which somewhere a stable minimum disappears so that the system settles to an equilibrium away from it, is called a **catastrophe**.

22.7. In order to visualize a catastrophe, we draw the graphs of the function $f_c(x)$ for various parameters c and look at the local minima. At a parameter value, where the number of local minima changes in some region, is called a catastrophe.





22.8. A **bifurcation diagram** displays the equilibrium points as they change in dependence of the parameter c. The vertical axes is the parameter c, the horizontal axes is x. At the bottom for $c = 0$, there are three equilibrium points, two local minima and one local maximum. At the top for $c = 1$ we have only one local minimum. Here is an important principle:

Catastrophes often lead to a strict and abrupt decrease of the minimal critical value. It is not possible to reverse the process in general.

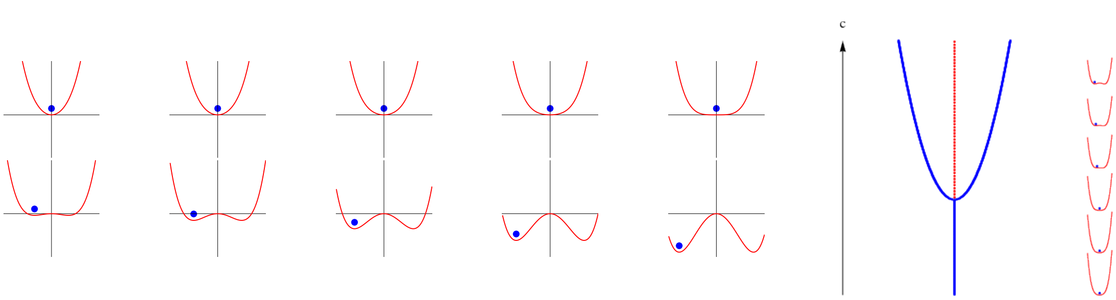
22.9. Let us look at this "movie" of graphs and run it backwards. By the same principle we do not end up at the position we started with. The new equilibrium remains the equilibrium nearby.

Catastrophes are in general **irreversible**.

22.10. We know this from experience: it is easy to screw up a relationship, reputation, get sick, have a ligament torn or lose somebody's trust. Building up a relationship, getting healthy or gaining trust usually happens continuously and slowly. Ruining the economy of a country or a company or losing a good reputation of a brand can be quick. It takes time to regain it.

Local minima can change discontinuously, when a parameter is changed. This can happen with perfectly smooth functions and smooth parameter changes.

We look at the example $f(x) = x^4 - cx^2$ with $-1 \leq c \leq 1$ in class.



Homework: Due Mar 25/2024

In this homework, we study a catastrophe for the function $f(x) = x^6 - x^4 + cx^2$, where c is a parameter between 0 and 1.

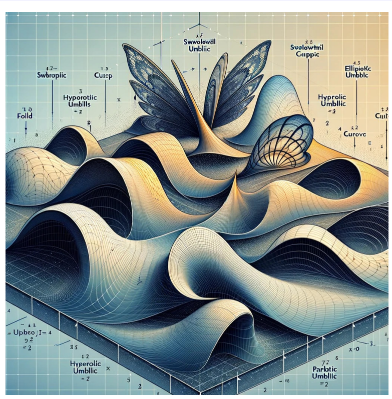
Problem 22.1: a) Find all the critical points in the case $c = 0$ and analyze their stability.
b) Find all the critical points in the case $c = 1$ and analyze their stability.

Problem 22.2: Plot the graph of the function $f(x)$ for $5 - 10$ values of c between 0 and 1. You can use desmos or Wolfram alpha. Mathematica example code is given in class.

Problem 22.3: If you change from $c = -1$ to 1, pinpoint the c value for which catastrophe (a discontinuous change of the minimum) occurs.

Problem 22.4: If you change back from $c = 0.6$ to -0.3, pinpoint the value for the catastrophe occurs. It will be different from the one in the previous question.

Problem 22.5: a) Write a 1000 word essay about "catastrophes" in math. In this problem you are officially allowed (and encouraged) to use AI. Be careful with prompting! Please also include what AI system you use and what your prompt was.
b) Repeat the exercise with a "one sentence definition". Again also include your prompt and the system that was used.
c) Have AI create an image about "catastrophe theory". Also here, be smart with prompting. You can use any tools. Include your picture and the tool you were using.



Here is a picture generated by chat GPT. It addresses more higher dimensional aspects of the theory in the context of singularity theory. That would be ok. Try to generate a picture which is closer to what we do here.

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INTRODUCTION TO CALCULUS

MATH 1A

Unit 23: Riemann integrals

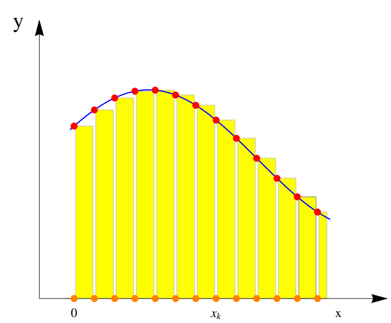
23.1. In this lecture, we define the **definite integral** $\int_a^b f(t)\,dt$ if f is a differentiable function. It has an interpretation as an **area under the curve**. Define $x_k = a + k\Delta x$ where $k = 0, \dots, n-1$ and $\Delta x = (b-a)/n$. The sum

$$S_n f = [f(x_0) + \cdots + f(x_{n-1})]\Delta x$$

is called a **Riemann sum**. It is a sum of areas of small rectangles of width Δx and height $f(x_k)$. It is a "left Riemann sum" because we evaluate the function to the left of the intervals.

Definition: Define

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x\ .$$



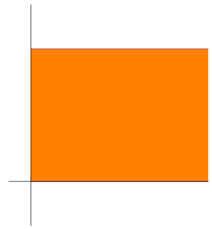
23.2. A very important result is that

For any differentiable function, the limit exists.

We can explicitly estimate the error: there are n little pieces where the region differs from the rectangle union. Each of these pieces has area $\leq M/n$, where M is the maximal slope that f can have in the given interval.

For non-negative f, the value $\int_0^x f(x)\,dx$ is the **area between the x-axis and the graph** of f. For general f, it is a **signed area**, the difference between two areas.

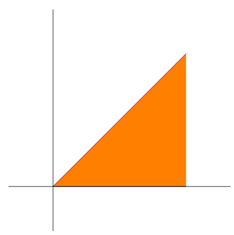
If $f(x) = c$ is constant, then $\int_0^x f(t)\,dt = cx$.



Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Adding up the Riemann sum is more difficult. Let k be the largest integer smaller than $xn = x/h$. Then

$$S_n f(x) = \frac{1}{n}\sum_{j=1}^{k} \frac{cj}{n} = \frac{ck(k+1)/2}{n^2} \; .$$

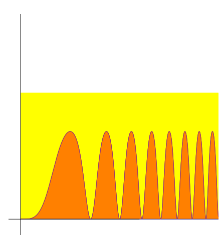
Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\,dt = cx^2/2$.



Linearity of the integral $\int_a^b f(t) + g(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$ and $\int_a^b \lambda f(t)\,dt = \lambda \int_a^b f(t)\,dt$.

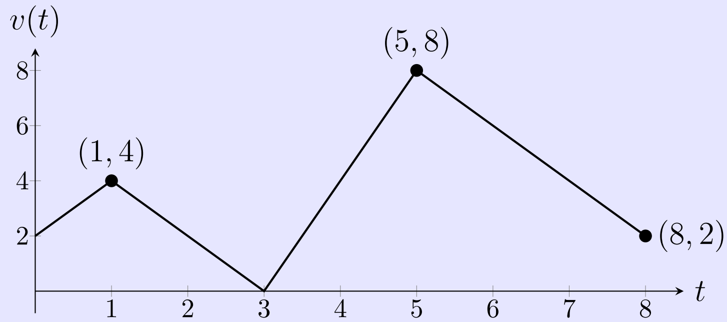
Upper bound: If $0 \leq f(x) \leq M$ for all x, then $\int_a^b f(t)\,dt \leq M(b-a)$.

$\int_0^x \sin^2(\sin(\sin(t)))/x \; dt \leq x$. **Solution**. The function $f(t)$ inside the interval is non-negative and smaller or equal to 1. The graph of f is therefore contained in a rectangle of width x and height 1.



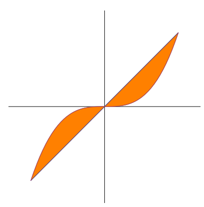
Homework due 3/27/2024

Problem 23.1: Below is the graph of the velocity of a bee traveling from a clover to a hive. Find the exact distance traveled by the bee between $t = 1$ and $t = 8$.



Problem 23.2: Lets look at the function $f(x) = \sin(x)$ on $[0, \pi]$.
a) Approximate the integral $\int_0^{\pi/2} \sin(x)\, dx$ using a Riemann sum with $\Delta x = \pi/4$.
b) Approximate the integral $\int_0^{\pi/2} \sin(x)\, dx$ using the Riemann sum with $\Delta x = \pi/6$.

Problem 23.3: The region enclosed by the graph of x and the graph of x^5 has a propeller type shape. Approximate its area by a Riemann sum using a Riemann sum with $\Delta x = 1/4$. It is your job to find a, b and n as well as the points $x_k = a + k(b-a)/n$.



Problem 23.4: Explain each rules with a picture:

- $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$.
- $\int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b f(x) - g(x)\, dx$.
- $\int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx$.

Problem 23.5: In this problem, it is crucial that you plot the function first. Split the integral up into parts. Find $\int_{-1}^4 f(x)\, dx$ for $f(x) = |x - |x - 2||$. As in 23.1 do not use a Riemann sum here. You can compute the value exactly.

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Unit 24: Fundamental theorem

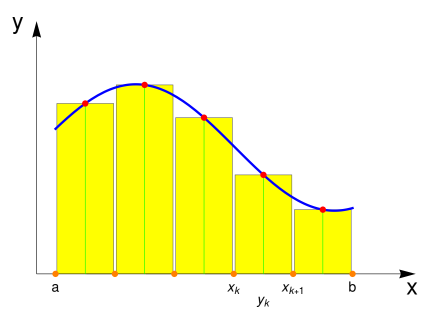
24.1. The **fundamental theorem of calculus** for differentiable functions allows to compute many integrals nicely without having to evaluate nasty and messy sums. First note that in the Riemann sum, we can chose any $y_k \in [x_k, x_{k+1}]$ to get the limit. The case $y_k = x_k$ is called the **left Riemann sum**, the case $y_k = x_{k+1}$ is called the **right Riemann sum**. The following result is also called the "evaluation part of the "fundamental theorem". It is the version is used most:

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Proof. By the **mean value theorem**, there exists in every interval $[x_k, x_{k+1}]$ a number y_k such that $f'(y_k) = (f(x_{k+1}) - f(x_k))/\Delta x$, because $\Delta x = x_{k+1} - x_k$. Now

$$\sum_{k=0}^{n-1} f'(y_k)\Delta x = \sum_{k=0}^{n-1} f(x_{k+1}) - f(x_k) = f(b) - f(a) .$$

Taking the limit $n \to \infty$ gives immediately $\int_a^b f'(x)\, dx = f(b) - f(a)$.



24.2. Why is this theorem so nice? One reason is that computing Riemann sums can be difficult. When trying to compute the area under the parabola, for example, we needed to sum up a difficult sum (as we have seen last time in class):

$$\int_0^1 x^2\, dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} (\frac{k}{n})^2 \frac{1}{n} = \lim_{n\to\infty} \frac{(2n-1)(n-1)}{6n^2} = \frac{1}{3} .$$

But that is insane. We needed to look up a formula for the sum of the first n squares. For $\int_0^1 x^4\, dx$ we would have to sniff out the formula $\lim_{n\to\infty} \frac{(n-1)(2n-1)\left(3n^2-3n-1\right)}{30n^5}$. We don't want to to do that! [1]

24.3. How do we use this theorem? Given an integral of a function, we ask our-self whether we know a function f such that its derivative is the function we have. We then write $\boxed{\int_a^b f'(x)\, dx = f(x)|_a^b = f(b) - f(a)}$. Notice the intermediate step where we write down the function $f(x)$ and the same bounds from the integral along a vertical bar. For now, we just stare at f' and see whether we can find a function f which has f' as its derivative. Knowing some derivatives is key.

$\int_1^2 x^2\, dx$. We know that x^2 is the derivative of $x^3/3$. So $\int_1^2 x^2\, dx = \frac{x^3}{3}|_1^2 = 7/3$.

$\int_0^\pi \sin(x)\; dx$. We know that $\sin(x)$ is the derivative of $-\cos(x)$. So $\int_0^\pi \sin(x)\; dx = -\cos(x)|_0^\pi = 2$.

$\int_0^5 x^7\, dx = \frac{x^8}{8}|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

$\int_0^{\pi/2} \cos(x)\; dx = \sin(x)|_0^{\pi/2} = 1$.

Find $\int_0^\pi \sin(x)\; dx$. **Solution**: The answer is 2.

For $\int_0^2 \cos(t+1)\; dt = \sin(x+1)|_0^2 = \sin(2) - \sin(1)$, the additional term $+1$ does not make matter as when using the chain rule, it goes away.

For $\int_{\pi/6}^{\pi/4} \cot(x)\; dx$, the anti-derivative is difficult to spot. It becomes only accessible if we know, where to look: the function $\ln(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\ln(\sin(x))|_{\pi/6}^{\pi/4} = \ln(\sin(\pi/4)) - \ln(\sin(\pi/6)) = \ln(1/\sqrt{2}) - \ln(1/2) = -\ln(2)/2 + \ln(2) = \ln(2)/2$.

[1] You could although like with Sum[$(k/n)^4/n, \{k, 0, n-1\}$]$/n$ or by asking any AI. Such formulas are well known in the literature and so no problem for any AI. They don't feel any pain when they are asked to do it for x^{100}. Or do they? Maybe they are just still too polite. In 20 years, they might punish you with a few days of misinformation and there is nothing you can do about it!

Homework: Due Mar 29/2024

Problem 24.1: Find a function f such that f' is what you see inside, then integrate
a) $\int_{-1}^{1} 4x^3 + 30x^2 \, dx$.
b) $\int_0^1 (x+1)^5 \, dx$.

Problem 24.2: Evaluate the following integrals:
a) $\int_2^3 5/(x-1) \, dx$, b) $\int_0^{\sqrt{\pi}} \sin(x^2) 4x \, dx$

Problem 24.3: Evaluate the following integrals:
a) $\int_1^2 2^x \, dx$,
b) $\int_0^{\sqrt{3}} \frac{1}{1+x^2} \, dx$,

Problem 24.4: Also here, in each case, just guess what derivative the integrand is.
a) $\int_0^{\sqrt{(\pi)}} \sin(x^2) x \, dx$
b) $\int_0^3 (3/2)\sqrt{1+x} \, dx$
c) $\int_0^{\sqrt{\ln(2)}} 4xe^{-x^2} \, dx$
d) $\int_e^{e^2} \frac{5}{x \ln(x)} \, dx$ e) $\int_0^1 \cos \sin \sin(x) \cos(\sin(x)) \cos(x) \, dx$

Problem 24.5: In this problem, we look at a situation where x appears in the bound.
a) Compute $F(x) = \int_0^{x^3} \sin(t) \, dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \, dt$ then find $G'(x)$

About the notation in 24.5 Note that writing $\int_0^{x^3} \sin(x) \, dx$ would have been **ill defined**. Computer scientists call such things "notation overload". It is commonly done, but confusing. Many programming languages allow to do that, but making use of it is a common source for programming errors. Mathematica for example allows to do it. The expression Integrate$[Sin[x], \{x, 0, x^3\}]$ gives the right answer. It internally rewrites this as Integrate$[Sin[t], \{t, 0, x^3\}]$. But the AI is well aware that it must have been a greenhorn who has entered the input. It is polite and does not tell you what it thinks about you. It might add you to an internal "fools" database. And again, there is nothing you can do about it.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024

Unit 25: Anti derivatives

25.1. The following statement follows from the fundamental theorem: take $a = 0, b = x$ and differentiate $\int_0^x f'(t)\, dt = f(x) - f(0)$ with respect to x. We get $\frac{d}{dx}\int_0^x f'(t)\, dt = f'(x)$. Replacing f' with f tells us:

$$\frac{d}{dx}\int_0^x f(t)\, dt = f(x)$$

Here is a geometric picture: pick a small h and $y \in [x, x+h]$ so that $f(y)h$ is $\int_x^{x+h} f(t)dt$

$$[\int_0^{x+h} f(t)\, dt - \int_0^x f(t)\, dt]\frac{1}{h} = \int_x^{x+h} f(t)dt\frac{1}{h} = f(y)h\frac{1}{h} = f(y)\ .$$

Taking the limit $h \to 0$ gives the statement of the fundamental theorem. [1]

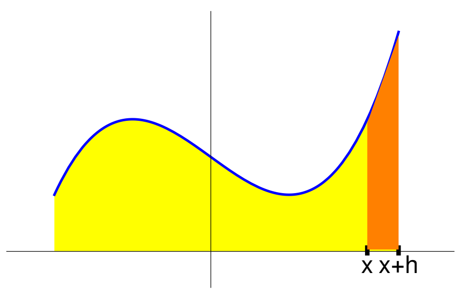


Figure 1. The fundamental theorem.

25.2. A function F with the property that $F'(x) = f(x)$ is called an **anti-derivative** of f. It is not unique but $F(x) = \int_0^x f(t)\, dt$ gives one anti-derivative. Define $\int_b^a f(x)dx = -\int_a^b f(x)\, dx$. This allows to define $F(x)$ for all x, also $x < 0$. Here is some more notation: we write an **indefinite integral** with a bare integral sign:

$$\int f(x)\ ; dx = F(x) + C\ ,$$

[1] This argument could be modified to see that the Riemann integral and the fundamental theorem works even for continuous functions.

if $F(x)$ is an anti-derivative. The reason for adding the constant C is that the anti-derivative is not unique. We would write for example $\int \sin(x)dx = -\cos(x) + C$. Note that the constant will disappear if we compute a **definite integral**

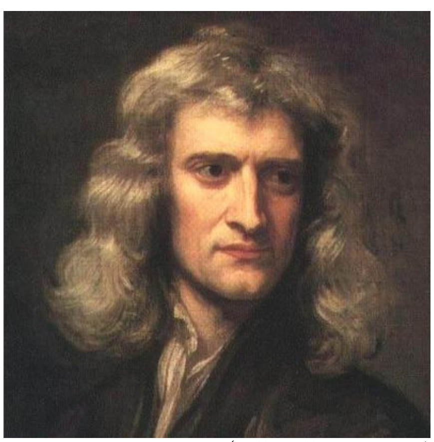
$$\int_a^b f(x) = F(b) - F(a)$$

if F is any anti-derivative. The **indefinite integral** is made up of some anti-derivative plus a constant.

25.3. You should have as many **anti-derivatives** "hard wired" in your brain. It really helps. Here are the core functions you should know.

function	anti derivative
1	x
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{cx}	$\frac{e^{cx}}{c}$
$\cos(cx)$	$\frac{\sin(cx)}{c}$
$\sin(cx)$	$\frac{-\cos(cx)}{c}$
$\tan(cx)$	$\ln(\cos(cx))/c$
$\frac{1}{x}$	$\ln(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\ln(x)$	$x\ln(x) - x$

25.4. Make your own table!




Newton ("Sir Slightly Annoyed") and Leibniz ("Mr Sour Face")

Isaac Newton and **Gottfried Leibniz** are the main figures for the fundamental theorem of calculus.

Homework: Due Apr 1/2024

Problem 25.1: Solve the following indefinite integrals:
a) $\int \sin(\sin(x)) \cos(x) \; dx$.
b) $\int \csc(x) \sec(x) dx$.
(rewrite $\csc(x) = 1/\sin(x)$, $\sec(x) = 1/\cos(x)$ so that the integrand is $\int \tan(x) \frac{1}{\cos^2(x)} \; dx$).

Problem 25.2: Find anti derivatives of the following functions:
a) $1/(1+x)$
b) $1/(1+x^2)$
c) $x/(1+x^2)$
d) $x^2/(1+x^2)$ (will appear in class today)
e) $x^4/(1+x^2)$ (will appear in class today)

Problem 25.3: Evaluate the following indefinite integrals:
a) $\int 3^x + x^3 \; dx$
b) $\int 3^{3^x} 3^x \; dx$

Problem 25.4: See whether you can write down the anti-derivatives in 5 minutes:
a) $\ln(x)/x$
b) $1/\sin^2(3x)$
c) $\frac{1}{\cos(\ln(x)))^2} \frac{1}{x}$
d) $\tan(1+2x)$
e) $\frac{\cos(x)}{2\sqrt{\sin(x)}}$

Problem 25.5: a) **A clever integral:** Evaluate the following integral (just by being clever, there is no algebra, and no work is needed):

$$\int_{-\pi}^{\pi} \sin(\sin(\sin(\sin(\sin(x))))) \; dx \; .$$

b) **An evil integral**: Evaluate $\int_{e^e}^{e^{e^e}} \frac{1}{\ln(\ln(x)) \ln(x) x} \; dx$.

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024

Unit 26: Review

Important results

Chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.

Implicit differentiation: $f(x, y) = c$ allows to compute y' if x, y are given

Extremal value theorem: a continuous function on $[a, b]$ has a max and min.

Intermediate value thm: continuous f on $[a, b]$ with $f(a) * f(b) < 0$ have roots.

Related rates: a rule relating $x(t)$ and $y(t)$ determines $y'(t)$ if $x'(t)$ is given.

Mean value theorem: a differentiable f on $[a, b]$ has x with $f'(x) = \frac{f(b)-f(a)}{b-a}$

Rolle's theorem: a differentiable f on $[a, b]$ with $f(a) = f(b)$ has a critical point.

Newton step: The step $T(x) = x - f(x)/f'(x)$ allows to get closer to a root of f.

Catastrophes: parameter values c, where the number of minima decreases.

Definite integrals: $\int_a^b f(x)\, dx$ is defined as a limit of Riemann sums.

Anti derivative: $F(x) = \int_0^x f'(t)\, dt$ satisfying $F' = f$.

Indefinite integral: given by $F + C$, where C is a constant and $F' = f$.

Fundamental theorem of calculus: $\int_a^b f'(x)\, dx = f(b) - f(a)$.

Fundamental theorem of calculus: $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$.

Signed area: $\int_a^b f(x)\, dx$ area between graph and x-axes, area below counted negative.

Important integrals

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \frac{1}{x} = \ln(x) + C$$

$$\int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int \frac{1}{\cos^2(x)} \, dx = \tan(x) + C$$

$$\int \frac{-1}{\sin^2(x)} \, dx = \cot(x) + C$$

$$\int \frac{1}{1+x^2} \, dx = \arctan(x) + C$$

$$\int \tan(x) \, dx = -\ln(\cos(x)) + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \, dx = \operatorname{arcsinh}(x) + C$$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 27: Sigmoid function

27.1. What is $\frac{d}{dx}\int_0^x f(t)\,dt$? If F is an anti-derivative, this is $\frac{d}{dx}[F(x)-F(0)] = f(x)$. [1] There are two aspects of the **fundamental theorem**:

$$\int_0^x f'(t)\,dt = f(x) - f(0), \qquad \frac{d}{dx}\int_0^x f(t)\,dt = f(x)\,.$$

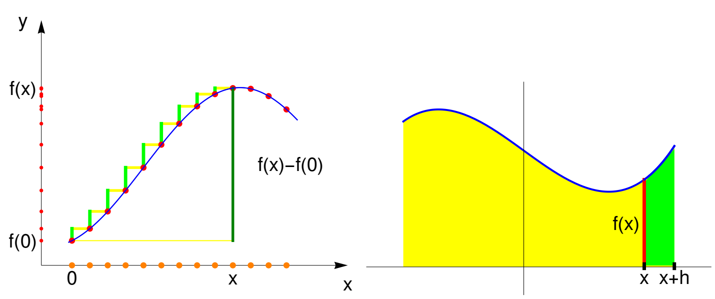


FIGURE 1. Integrate the derivative or differentiate the integral.

27.2. The **activation function** for **neural networks** is given by a differentiable function like $\sigma(x) = (\tanh(x/2)+1)/2 = e^x/(1+e^x)$ rather than a **step function** $(\text{sign}(x)+1)/2$. The first one is the **sigmoid function**. You work on this a bit in this homework.

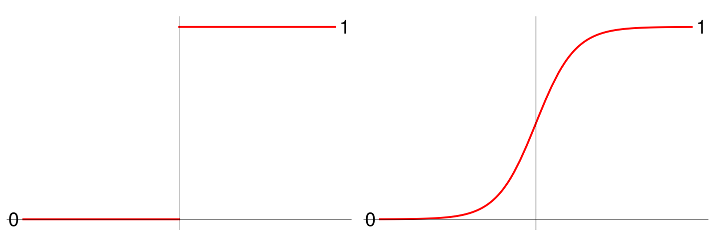


FIGURE 2. The **step function** $(\text{sign}(x)+1)/2$ is non-differentiable, the **sigmoid function** $(\tanh(x/2)+1)/2 = e^x/(1+e^x)$ is differentiable.

The reason is that differentiability allows to use gradient descent minimum algorithms (GDM) similarly as the Newton method we have seen to find maxima or minima. Sometimes one sees $\sigma(x) = \frac{1}{1+e^{-x}}$. Why is this the same?

[1] All except one student got this wrong in the exam.

Homework: due April 8, 2024

Problem 27.1: Verify $\sigma(x) = \frac{\tanh(x/2)+1}{2}$, where $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ and $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x+e^{-x}}{2}$.

Problem 27.2: In this problem we work on the **logistic distribution** in statistics.
a) Check that $F(x) = (\tanh(\frac{x}{2}) + 1)/2$ (which by 27.1) is $\sigma(x)$) has the derivative

$$f(x) = \frac{1}{4\cosh^2(\frac{x}{2})} \; .$$

It is called the **logistic distribution**.
b) Why is $\int_{-\infty}^{\infty} f(x)\, dx = 1$? Hint. $\int_{-a}^{a} f(x)\, dx = F(a) - F(-a) = 2\tanh(x/2)$.

Problem 27.3: The function $G(x) = \frac{\arctan(x)}{\pi} + \frac{1}{2}$ resembles $F(x) = \sigma(x)$. Plot both $f = F'$ and $g = G'$ (2 points) then complete the following table (2 points each):

$\frac{d}{dx}\int_0^x \frac{1}{4\cosh^2(\frac{t}{2})}\, dt = \boxed{}$	$\int_0^x \frac{1}{4\cosh^2(\frac{t}{2})}\, dt = \boxed{}$
$\frac{d}{dx}\int_0^x \frac{1}{\pi(1+t^2)}\, dt = \boxed{}$	$\int_0^x \frac{1}{\pi(1+t^2)}\, dt = \boxed{}$

Problem 27.4: The sigmoid function $F(x)$ is also called the **standard logistic function** because it satisfies the **logistic equation** $F'(x) = F(x)(1 - F(x))$. Verify this. (Compute both $F'(x)$ and $F(x)(1 - F(x))$ and compare).

Problem 27.5: The function $N(x) = (1 + \text{erf}(x))/2$ looks similar to the sigmoid function. The error function erf satisfies $\text{erf}'(x) = 2e^{-x^2}/\sqrt{\pi}$.
a) Plot $n(x) = \text{erf}'(x)$.
b) Which of the $f(x), g(x), n(x)$ is the tallest at $x = 0$?
c) Which of the $f(x), g(x), n(x)$ is the tallest at $x = 10$?

Oliver Knill, knill@math.harvard.edu, Math 1a, Spring, 2024

[2] This is extremely important in machine learning as the derivative is given in terms of the same function. One uses this in backpropagation.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 28: Substitution

28.1. Functions like e^{6x} or $1/(1+x)$ have been integrated by seeing $\boxed{\int f(cx+a)\,dx = F(cx+a)/c}$, if $F' = f$. We have also seen how to "spot the chain rule": if $f(x) = g(u(x))u'(x)$, $\boxed{\int f(x)\,dx = G(u(x)) + C}$, where $G' = g$.

28.2. Example: $\int e^{x^4+x^2}(4x^3+2x)\,dx = e^{x^4+x^2} + C$.
Example: $\int \sqrt{x^5+1}x^4\,dx = (2/15)(x^5+1)^{3/2}$.
Example: $\int \frac{\ln(x)}{x}\,dx = \ln(x)^2/2 + C$

28.3. The **method of substitution** formalizes this and makes it more systematic: **A)** select part of the formula, call it u. **B)** write $du = u'dx$ and **C)** replace dx with du/u'. **D)** If all terms x have disappeared, integrate. **E)** Back substitute the variable x. If things should not work, go back to A) and try an other u.

$$\int f(\text{u(x)})\,\text{u'(x)} \quad \text{dx} = \int g(\text{u})\,\text{du}\,.$$

28.4. Example: to get $\int \ln(x)/x\,dx$, pick $u = \ln(x)$, compute $du = (1/x)dx$ and so $dx = xdu$. We get $\int udu = u^2/2 + C$. Back substitution gives $\ln^2(x)/2 + C$.
Example: to get $\int \ln(\ln(x))\frac{1}{\ln(x)x}\,dx$, try $u = \ln(x)$ and $du = (1/x)dx$, then plug this in to get $\int \ln(u)/u\,du = \ln^2(u)/2 + C$. Back substitute to get $\ln^2(\ln(x)) + C$.

Example: To get $\int \frac{x}{1+x^4}\,dx$, substitute $u = x^2, du = 2xdx$ to get gives $(1/2)\int du/(1+u^2)\,du = (1/2)\arctan(u) = (1/2)\arctan(x^2) + C$.

Example: To get $\sin(\sqrt{x})/\sqrt{x}$, try $u = \sqrt{x}, x = u^2, dx = 2udu$. The result is $-2\cos(\sqrt(x)) + C$.

28.5. Example: (harder) $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$ might trigger the reflex $u = \sqrt{x^2+1}$ but this does not work does not work. With $u = x^2+1$ and $du = 2xdx$ and $dx = du/(2\sqrt{u-1})$. We get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}}\,du = \int \frac{(u-1)}{2\sqrt{u}} = \int \frac{u^{1/2}}{2} - \frac{u^{-1/2}}{2}\,du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2}\,.$$

28.6. For **definite integrals** $\int_a^b f(x)\,dx$, one could find an anti-derivative as described and fill in the bounds.

$$\int_a^b g(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} g(u)\,du\;.$$

Proof. The right hand side is $G(u(b))-G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has an anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b))-G(u(a))$.

Example: $\int_0^1 \frac{1}{5x+1}\,dx = [\ln(u)]/5|_1^6 = \ln(6)/5$.
Example: $\int_3^5 \exp(4x-10)\,dx = [\exp(10)-\exp(2)]/4$.

Homework

Problem 28.1: Find the following anti-derivatives. But do it formally following the steps, even if you see the answer already.
a) $\int x^2\sin(x^3)\,dx$ b) $\int e^{x^6+x}(6x^5+1)\,dx$
c) $\int -\cos(\sin(3x))\cos(3x)/3\,dx$ d) $\int e^{\tan(2x)}/\cos^2(2x)\,dx$.

Problem 28.2: Compute the following definite integrals. It is fine to find first an anti-derivative and only at the end place the bounds:
a) $\int_1^2 \sqrt{x^5+x}(5x^4+1)\,dx$ b) $\int_0^{\sqrt{\pi}} 6\sin(x^2)x\,dx$.
c) $\int_e^{e^2} \frac{\sqrt{\ln(x)}}{x}\,dx$ d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}}\,dx$.

Problem 28.3: Compute $\int_e^{2e} \frac{dx}{\sqrt{\ln(x)}x}$.

Problem 28.4: Find the indefinite integrals:
a) $\int \frac{x^5}{\sqrt{x^2+1}}\,dx$. b) $\int \frac{1}{x(1+\ln(x)^2)}\,dx$

Problem 28.5: Find the anti-derivatives of a) $\frac{\cos(x^3)}{e^{\sin(x^3)}}x^2$ b) $\cot(\sqrt{x})/\sqrt{x}$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 29: Integration by parts

29.1. Integration by parts is based on the **product rule** $(uv)' = u'v + uv'$. It complements the method of substitution we have seen last time and which had been reversing the **chain rule**. As a rule of thumb, always try first to **1) simplify a function and to integrate using known functions**, then **2) try substitution** and finally **3) try integration by parts**.

$$\int \text{u(x)}\ \text{v'}\,(x)dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

29.2. Lets try this with $\int x\sin(x)\,dx$. First identify what you want to differentiate and call it u, the part to integrate is called v'. Now, write down uv and subtract a new integral which integrates $u'v$:

$$\int \text{x}\ \sin(x)\ dx = \text{x}\ (-\cos(x)) - \int 1\ (-\cos(x)\ )\,dx = -x\cos(x) + \sin(x) + C\ dx\ .$$

On paper, we can stream-line this by just placing an **down-arrow** under the expression you differentiate and an **up-arrow** under the expression you integrate. You remember to first integrate, then subtract the integral of the expression where you both integrate and differentiate. If you like to write down the u, v's, do so. What you need to remember is $\boxed{\int udv = uv - \int vdu}$.

29.3. Find $\int xe^x\,dx$. **Solution.** You want to differentiate x and integrate e^x.

$$\int \text{x}\ \exp(x)\ dx = x\exp(x) - \int 1\cdot\exp(x)\,dx = x\exp(x) - \exp(x) + C\ dx\ .$$

29.4. Find $\int \ln(x)\,dx$. **Solution.** While there is only one function here, we need two to use the method. Let us look at $\ln(x)\cdot 1$:

$$\int \ln(x)\ 1\ dx = \ln(x)x - \int \frac{1}{x}x\,dx = x\ln(x) - x + C\ .$$

29.5. Find $\int x\ln(x)\,dx$. **Solution.** Since we know from the previous problem how to integrate ln we could proceed by taking $x = u$. We can also take $u = \ln(x)$ and $dv = x$:

$$\int \ln(x)\ \text{x}\ dx = \ln(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\,dx$$

which is $\ln(x)x^2/2 - x^2/4$.

29.6. We saw that it is good to differentiate ln's. The word LIATE (explained below) tells which functions we want to call u and differentiate.

Marry go round: Find $I = \int \sin(x)\exp(x)\ dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\ dx\ .$$

Lets do it again:

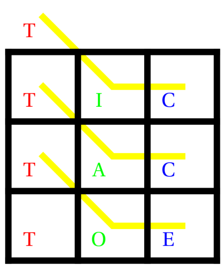
$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\ dx.$$

We moved in circles and are stuck! But not really! We have derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

which we can solve for I and get $\boxed{I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2.}$

Tic-Tac-Toe



Integration by parts can become complicated if we need to repeat it several times. Keeping the order of the signs can be especially daunting. Fortunately, there is a powerful **tabular integration by parts method**. It has been called **"Tic-Tac-Toe"** in the movie Stand and deliver. Lets call it **Tic-Tac-Toe** therefore.

29.7. Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

$(x-1)^3$	$\exp(2x)$	
$3(x-1)^2$	$\exp(2x)/2$	$\oplus$
$6(x-1)$	$\exp(2x)/4$	$\ominus$
6	$\exp(2x)/8$	$\oplus$
0	$\exp(2x)/16$	$\ominus$

The anti-derivative is

$$(x-1)^3 e^{2x}/2 - 3(x-1)^2 e^{2x}/4 + 6(x-1)e^{2x}/8 - 6e^{2x}/16 + C\ .$$

29.8. Find the anti-derivative of $x^2\cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	$\oplus$
2	$-\cos(x)$	$\ominus$
0	$-\sin(x)$	$\oplus$

The anti-derivative is $x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$.

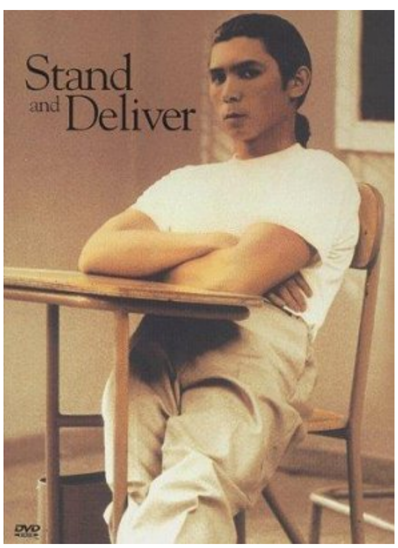
29.9. Find the anti-derivative of $x^7\cos(x)$. **Solution:**

x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	$\oplus$
$42x^5$	$-\cos(x)$	$\ominus$
$120x^4$	$-\sin(x)$	$\oplus$
$840x^3$	$\cos(x)$	$\ominus$
$2520x^2$	$\sin(x)$	$\oplus$
$5040x$	$-\cos(x)$	$\ominus$
5040	$-\sin(x)$	$\oplus$
0	$\cos(x)$	$\ominus$

The anti-derivative is

$$\begin{aligned} F(x) &= x^7\sin(x) + 7x^6\cos(x) - 42x^5\sin(x) \\ &- 210x^4\cos(x) + 840x^3\sin(x) + 2520x^2\cos(x) \\ &- 5040x\sin(x) - 5040\cos(x) + C\ . \end{aligned}$$

29.10. Do this without this method and you see the value of the method.
[1] [2] [3]:



I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to the article of V.N. Murty.

$$\begin{aligned} \int fg\,dx &= fg^{(-1)} - f^{(1)}g^{(-2)} + f^{(2)}g^{(-3)} - \dots \\ &- (-1)^n \int f^{(n+1)}g^{(-n-1)}\,dx \end{aligned}$$

The Tic-Tac-Toe method can be verified by induction because the f function is differentiated again and again and the g function is integrated again and again. The alternating minus-plus-signs come from the fact that we subtract an integral and so change the sign of both. We always pair a k'th derivative with a $k+1$'th integral and take the sign $(-1)^k$.

Coffee or Tea?

[1]V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, p. 90-94.
[2]D. Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, p. 307-311.
[3]K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, p. 542-543

29.11. We want to first differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to ”lipton” (the tea).

For coffee lovers, there is **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials. Now, **LIATE** is close to ”latte” (the coffee).

Whether you prefer to remember it as a **“coffee latte”** or a **“lipton tea”** is up to you.

If you should not like neither coffee, nor tea, there is “opportunist method”:

Just integrate what you can integrate and differentiate the rest.

An don’t forget to consider integrating 1, if nothing else works.

Homework

Problem 29.1: Integrate $\int x^3 \ln(x)\, dx$.

Problem 29.2: Integrate $\int x^5 \sin(x)\, dx$

Problem 29.3: Find the anti derivative of $\int 2x^6 \exp(x)\, dx$.

Problem 29.4: Find the anti derivative of $\int \sqrt{x} \ln(x)\, dx$.

Problem 29.5: Find the anti derivative of $\int \sin(x) \exp(-x)\, dx$.

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INTRODUCTION TO CALCULUS

MATH 1A

Unit 30: Partial Fractions

30.1. The method of **partial fractions** allows to extend the range of integrals we can do. It is just about **algebra**. As we know how to integrate polynomials like x^4+5x+3 we would like to be able to integrate rational functions $f(x) = \frac{p(x)}{q(x)}$, where p, q are polynomials. As we know already the integrals of $1/(x+a)$ and $1/(1+x^2)$, why not try to write a general fraction using such terms.

30.2. First of all, we should be reminded on how to calculate with fractions. How do we add, subtract, multiply or divide fractions? What does it mean to take a fraction to a power or the exponential of a fraction?

Definition: The **partial fraction method** writes $p(x)/q(x)$ as a combination of functions of the form $A/(x+a)$, which we can integrate.

30.3.

In order to integrate $\int \frac{1}{(x-a)(x-b)}\,dx$, write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} .$$

cross multiply:

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)} .$$

and match the nominator:

$$1 = Ax - Ab + Bx - Ba .$$

This implies $A+B=0, Ab-Ba=1$ allowing to solve for A, B.

30.4. Example: To integrate $\int \frac{2}{1-x^2}\,dx$ we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \ln(1+x) - \ln(1-x) .$$

30.5. Example: Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} .$$

Now multiply out and solve for A, B:

$$A(x-2) + B(x-3) = 5 - 2x .$$

This gives the equations $A + B = -2, -2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. We have not obtained

$$\frac{5-2x}{x^2-5x+6} = -\frac{1}{x-3} - \frac{1}{x-2}$$

and can integrate:

$$\int \frac{5-2x}{x^2-5x+6}\, dx = -\ln(x-3) - \ln(x-2) .$$

30.6. Example: Integrate $f(x) = \int \frac{1}{1-4x^2}\, dx$. **Solution.** The denominator is factored as $(1-2x)(1+2x)$. Write

$$\frac{A}{1-2x} + \frac{B}{1+2x} = \frac{1}{1-4x^2} .$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x)\, dx = \frac{1}{4}\ln(1-2x) - \frac{1}{4}\ln(1+2x) + C .$$

30.7. The following **Hospital method** or **residue method** saves time especially with many functions where we would a complicated system of linear equations would have to be solved.

If a is different from b, then the coefficients A, B in

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} ,$$

are

$$A = \lim_{x\to a}(x-a)f(x) = p(a)/(a-b),\ B = \lim_{x\to b}(x-b)f(x) = p(b)/(b-a) .$$

The reason is: if we multiply the identity with $x - a$ we get

$$\frac{p(x)}{(x-b)} = A + \frac{B(x-a)}{x-b} .$$

Now we can take the limit $x \to a$ without peril and end up with $A = p(a)/(x-b)$.

30.8. Example: Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = 11/12$, and $B = \frac{2*(-8)+3}{(-8-4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8} .$$

The integral is

$$\frac{11}{12}\ln(x-4) + \frac{13}{12}\ln(x+8) .$$

30.9. Example: Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2}\ln(x-1) - 7\ln(x-2) + \frac{13}{2}\ln(x-3) .$$

30.10. A remark: it is always amazing to see how fast many students can do this, even with many variables. Once you realize the principle, the computation is very nice. Mathematically one has understood the "residue method" which is also important in complex analysis, which is calculus done when the real numbers are extended to complex numbers.

To get the constant A in the part $A/(x-a)$, just divide out the factor $x-a$ from the original expression for $f(x)$, then set $x = a$.

30.11. Example: Let

$$f(x) = \frac{1}{(x-3)(x-5)(x-11)(x-4)} = A/(x-3)+B/(x-5)+C/(x-11)+D/(x-4) .$$

To get the constant C for example, just delete the factor $(x-11)$ in the left part, then put $x = 11$:

$$C = \frac{1}{(x-3)(x-5)(x-4)} = \frac{1}{(11-3)(11-5)(11-4)} = 1/(8*6*7) .$$

Homework

Problem 30.1: Compute the following integrals by simplification. There is no partial fraction needed, you just should see how to simplify the expression.
a) $\int \frac{x+1}{x}\,dx$.
b) $\int \frac{x+1}{x^2-1}\,dx$.
c) $\int \frac{x^3+3x^2+3x+1}{x^2+2x+1}\,dx$.
d) $\int \frac{x^2-2x-3}{x^2-5x+6}\,dx$.
e) $\int \frac{x-1}{x+1}\,dx$.

Problem 30.2: Solve the following integrals using partial fraction
a) $\int \frac{1}{x^2-4}\,dx$.

b) $\int \frac{1}{(x-1)(x+1)(x-3)}\,dx$.

Problem 30.3: And now these two
a) $\int \frac{1}{x^2-14x+45}\,dx$

b) $\int \frac{2}{x^2-9}\,dx$

Problem 30.4: As this is the last lecture on integration, lets also review other methods. Solve the following integrals:
a) Compute $\int (x-1)^7 \sin(3x)\,dx$.

b) Compute $\int \frac{1}{(1+x^2)(1+\arctan(x)^2)}\,dx$.

Problem 30.5: For this example, you really need to use the Hospital method: $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)}\,dx$. Write the function as

$$f = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+7} + \frac{D}{x-3}$$

then figure out the constants A, B, C, D.

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INTRODUCTION TO CALCULUS

MATH 1A

Unit 31: Extrema review: Economics

31.1. This week we review a few themes taught in this course. First we recall how to find maxima and minima. In economics, you want to minimize or maximize a quantity. Calculus expresses this using critical points. Two theorems decide whether we have minima or maxima. In economics, parameters are often external, like supply and demand, the interest rates etc. Given these parameters, one wants to extremize a quantity, like profit of a company. We have also looked at what happens if parameters are deformed. Deformation parameters are called "weights" in "transformer networks".

31.2. Calculus plays a pivotal role in **economics**. Lets also grasp the opportunity to get acquainted with some jargon in economics. Economists talk differently: $f' > 0$ means **growth** or **boom**, $f' < 0$ means **decline** or **recession**, a vertical asymptote is a **crash**, a horizontal asymptote is a **stagnation**, a discontinuity is labled as **"inelastic behavior"**, the derivative of something is the **"marginal"** of it. An example is marginal revenue.

Definition: The **marginal cost** is the derivative of the **total cost**.

31.3. The marginal cost and total cost are functions of the quantity x of goods:
Example: Assume the total cost function is $C(x) = 10x - 0.01x^2$. Use the marginal cost in order to minimize the total cost. **Solution**. Differentiate $C' = 10 + 0.02x$ At a minimum, this derivative is zero. Here at $x = 50$.

Example: You sell spring water. The marginal cost to produce it at time x (years) is $f(x) = 1000 - 2000 \sin(x/6)$. For which x is the total cost maximal? **Solution**. We look for points where $F'(x) = f(x) = 0, F''(x) < 0$. This is the case for $x = \pi$. The cost is maximal in about 3 years.
In the book "Don't worry about Micro, 2008", Dominik Heckner and Tobias Kretschmer tell the following strawberry story (quoted in verbatim):

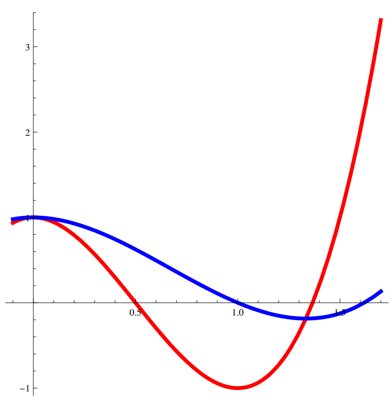
Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

31.4. If $F(x)$ is the **total cost function** in dependence of the quantity x, then $F' = f$ is called the **marginal cost**.

Definition: The function $g(x) = F(x)/x$ is called the **average cost**.

Definition: A point, where $f = g$ is called a **break-even point**.

If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points, where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



31.5. The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, lets call it the "strawberry theorem":

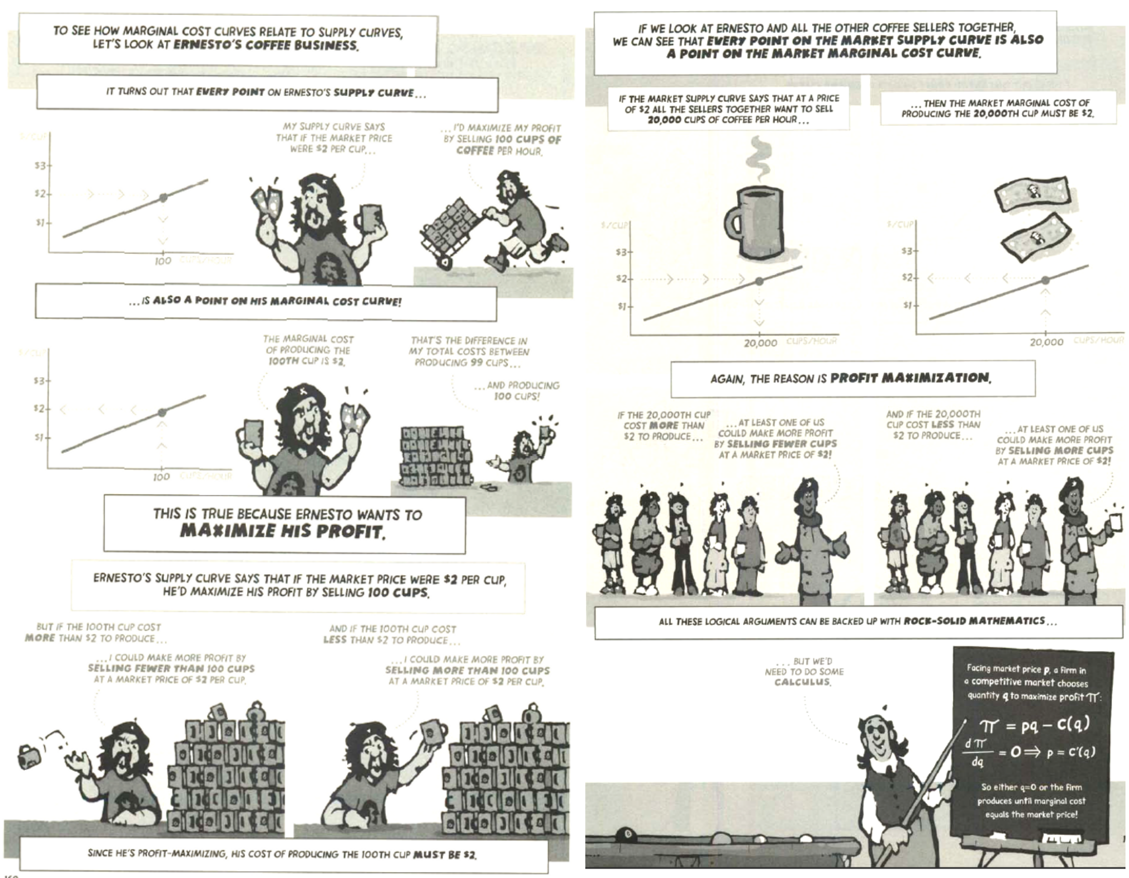
Strawberry theorem: $g'(x) = 0$ if an only if $f = g$.

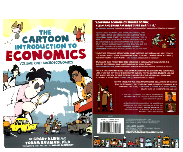
Proof. $g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g)$.

More extremization

31.6. Lets review also some extremization problems:
[**Example:** Find the **rhomboid** with side length 1 which has maximal area. Use angle α to extremize.

Find the ellipse of length $2a$ and width $2b$ which has fixed area $\pi ab = \pi$ and for which the sum of diameters $2a + 2b$ is maximal. **Solution.** Find $b = 1/a$ from the first equation and plug into the second equation.





Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang. You can detect the strawberry theorem ($g' = 0$ is equivalent to $f = g$) can be seen on the blackboard.

Homework

Problem 31.1: Find the break-even point for an economic system if the marginal cost is $f(x) = 1/x$.

Problem 31.2: Let $f(x) = \cos(x)$. Compute $F(x)$ and $g(x)$ and verify that $f = g$ agrees with $g' = 0$.

Problem 31.3: Time to review standard extremization: for smaller groups, production usually increases when adding more workforce. After some time, bottle necks occur, not all resources can be used at the same management and bureaucracy is added. We make a model to find the maximal production parameters. The **production function** in an office gives the production $Q(L)$ in dependence of labor L. Assume

$$Q(L) = 5000L^3 - 3L^5 .$$

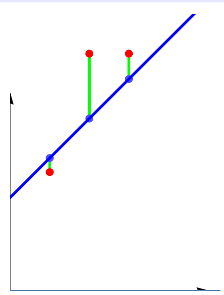
Find L which gives the maximal the production. Use the second derivative test.

Problem 31.4: **Marginal revenue** f is the rate of change in **total revenue** F. As total and marginal cost, these are functions of the **cost** x. Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the points, where the total revenue has a local maximum. Also here, use the second derivative test.

Problem 31.5: We do linear regression. Find the best line $y = mx + b$ through the points

$$(x_1, y_1) = (1, 3), (x_2, y_2) = (2, 6), (x_3, y_3) = (3, 6) .$$

To do so, first center the data by subtracting the average (2,5), then minimize $f(m) = \sum_{k=1}^{3}(mx_k - y_k)^2$ Now $m(x-2)+5$ is the best fit.

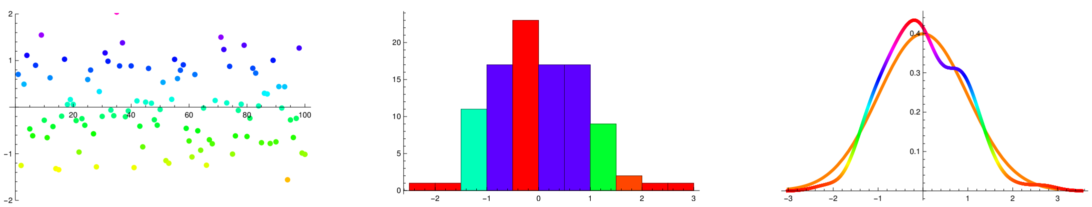


INTRODUCTION TO CALCULUS

MATH 1A

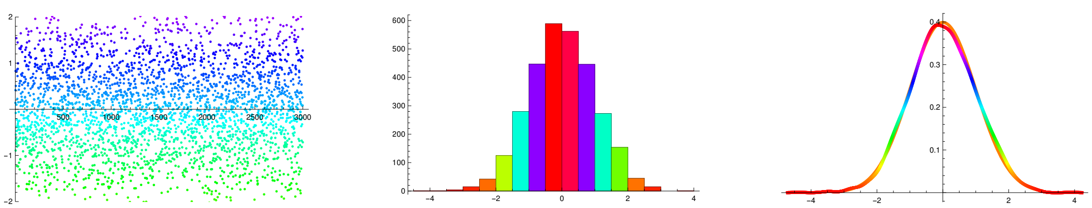
Unit 32: Statistics

32.1. Statistics describes data using functions called **probability distributions**. The topic allows us to review some integration. To see how one gets from data to functions, lets look at the following 100 data points. If we count, how many data values fall into a specific interval, we get a **histogram**. Smoothing this histogram and scaling so that the total integral is 1 produces a **probability distribution function**. This allows us to describe data, discrete sets of points with functions.



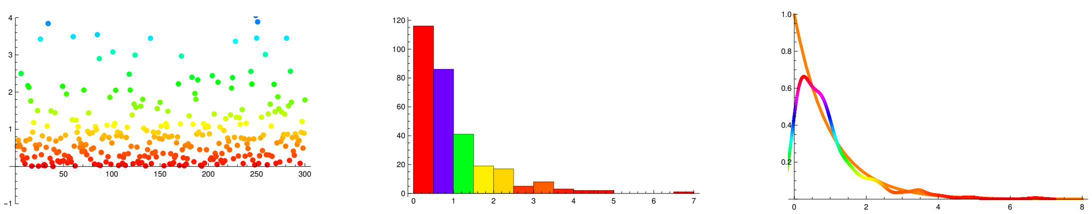
100 data points, the histogram and a smooth interpolation PDF.

32.2. If we take 3200 points, the data distribution has a bell curve shape. In the last picture, we also included the graph of $f(x) = e^{-x^2/2}/\sqrt{2\pi}$.

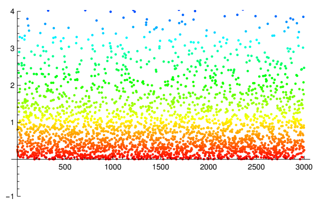


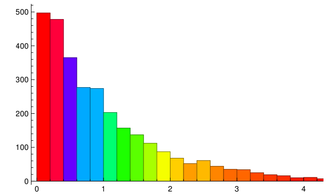
3200 data points, the histogram and a smooth interpolation PDF.

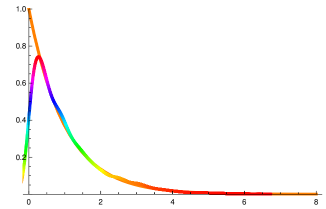
32.3. There are some data of "waiting times. These data are positive. We then draw the histogram and a smooth interpolation function. Lets do it again first for 320 data points and then for 3200 data points.



320 data points, the histogram and a smooth interpolation PDF







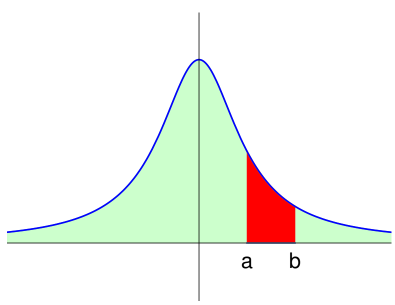
3200 other data points, the histogram and a smooth interpolation PDF

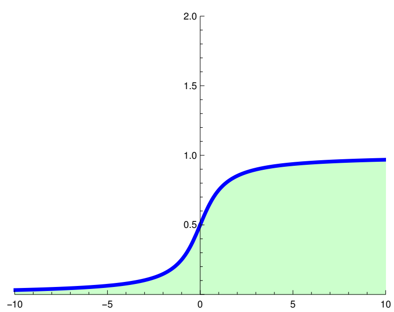
Definition: A **probability density function** is a piecewise non-negative continuous function f with the property $\int_{-\infty}^{\infty} f(x)\,dx = 1$. The anti derivative $F(x) = \int_{-\infty}^{x} f(t)\,dt$ is called the **cumulative distribution function**.

The cumulative distribution function is increasing from 0 to 1 as its derivative is non-negative.

Definition: **moments** of the PDF are integrals of the form

$$M_n = \int_{-\infty}^{\infty} x^n f(x)\,dx$$





PDF and CDF

32.4. For $n = 0$, we know $M_0 = 1$. The first moment M_1 is called the **expectation** or **average**:

Definition: The **expectation** of probability density function f is

$$m = \int_{-\infty}^{\infty} x f(x)\,dx\ .$$

32.5. The second moment allows us to get the **variance**:

Definition: The **variance** of probability density function f is

$$\mathrm{Var}(f) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - m^2\ ,$$

where m is the expectation. We can write $\mathrm{Var}(f) = M_2 - M_1^2$.

Definition: The integral $\mu_k = \int_{-\infty}^{\infty} (x-m)^k f(x)\,dx$ is called the k'th **central moment**. We have $\mu_2 = \mathrm{Var}$.

Definition: The square root σ of the variance is called the **standard deviation** It is the expected deviation from the mean.

32.6. From it, one can get the **normalized central moment** $C_k = \mu_k/\sigma^k$ which is $C_k = \int_{-\infty}^{\infty} (\frac{(x-m)}{\sigma})^k f(x)\, dx$. Computing moments, central moments and normalized central moments leads often to "integration by parts" problems:

The expectation of the geometric distribution $f(x) = e^{-x}$

$$\int xe^{-x}\, dx = 1 \ .$$

The variance of the geometric distribution $f(x) = e^{-x}$ is 1 and the standard deviation 1 too. To see this, let us compute

$$\int_0^{\infty} x^2 e^{-x}\, dx \ .$$

x^2	e^{-x}	
$2x$	$-e^{-x}$	$\oplus$
2	e^{-x}	$\ominus$
0	e^{-x}	$\oplus$

You have already computed the expectation of the standard Normal distribution $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-x^2/2}\, dx = 0 \ .$$

The variance of the standard Normal distribution $f(x)$ is $\frac{1}{\sqrt{2\pi}}$ times

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx \ .$$

We can compute this integral by partial integration too but we have to split it as $u = x$ and $v = xe^{-x^2/2}$.

$$-xe^{-x^2/2}|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi} \ .$$

The variance therefore is $\boxed{1}$.

32.7. Example: The distribution on $[-1, 1]$ with function $(1/\pi)(1-x^2)^{-1/2}$ there and 0 everywhere else is called the **arcsin-distribution**. What is the cumulative distribution function? What is the mean m? What is the standard deviation σ? As we will see in class, $m = 0, \sigma = 1/\sqrt{2}$.

Homework

Problem 32.1: The function $f(x) = \cos(x)/2$ on $[-\pi/2, \pi/2]$ is a probability density function. Its mean is 0. Find its variance $\int_{-\pi/2}^{\pi/2} x^2 \cos(x)\, dx/2$.

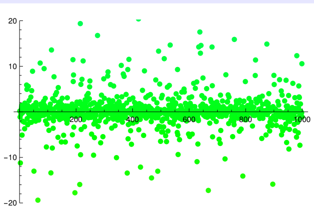
Problem 32.2: The **uniform distribution on** $[0,1]$ is a distribution with probability density function is $f(x) = 1$ for $0 \leq x \leq 1$ and 0 elsewhere. Compute: a) the n'th moment M_n, b) the variance $\mathrm{Var}[f] = M_2 - M_1^2$, and c) the standard deviation $\sigma = \sqrt{\mathrm{Var}[f]}$.

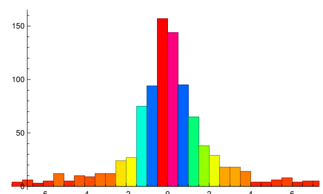
Problem 32.3: We have seen in lecture 27 that the **sigmoid function** $F(x) = (\tanh(\frac{x}{2}) + 1)/2$ has the derivative $f(x) = \frac{1}{4\cosh^2(\frac{x}{2})}$. It is called the **logistic distribution**. You have already checked that $\int_{-\infty}^{\infty} f(x)\, dx = 1$. Use both a computer algebra system as well as an AI tool to compute.
a) The variance $\int_{-\infty}^{\infty} x^2 f(x)\, dx$ and the entropy $-\int_{-\infty}^{\infty} f(x)\log(f(x))\, dx$

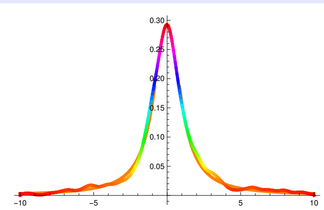
[1]

Problem 32.4: a) Verify again that the **Cauchy distribution** with PDF $f(x) = \frac{(1/\pi)}{x^2+1}$ has the CDF $F(y) = 1/2 + \arctan(y)/\pi$.
b) What can you say about the variance of this distribution?

Problem 32.5: If we take random numbers x_k in $[0,1]$ then $\tan(\pi x_k)$ are Cauchy distributed. a) What is the probability that such a random number hits $[-1,1]$? (Hint: it is $F(b) - F(a)$ where F is the function in 31.4).
b) Use a calculator (or your favorite program) to compute at least 20 Cauchy distributed random numbers. How many hit the interval $[-1,1]$? How many would you have expected to hit? (The picture below does that for a few hundred data). In Mathematica these numbers can be accessed by Tan[Pi $*$ Random[]].







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[1] In both cases, give the entry you entered into the computer algebra system like wolfram alpha and the prompt you used to extract the number from a tool like chat gpt 4.

Unit 33: Artificial Intelligence

Universal Approximation Theorem

33.1. Multi-layer feed forward networks need to be able to model large class of functions. The basic mathematical problem has been solved already in the 1980ies in arbitrary dimensions by Cybenko and Hornik. They proved the **universal approximation theorem**. Here, we look at it in one dimensions using the **sigmoid activation function**

$$\sigma(x) = 1/(1+e^{-x}) \ .$$

A **neural network** is a finite sum of functions of the form $c\sigma(ax+b)$, where a, b, c are constants.

Every continuous $f(x)$ on $[0,1]$ can be approximated by **neural networks**.

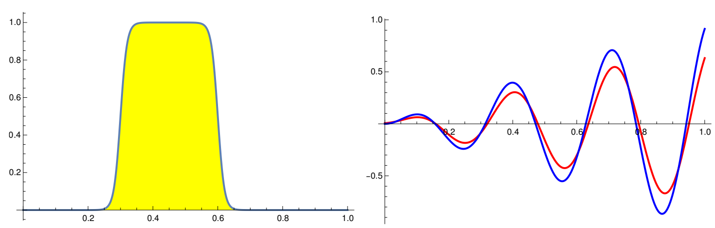

FIGURE 1. The neural network function $\sigma(100(x-0.3))-\sigma(100(x-0.6))$ is close to the function which is 1 on $[0.3, 0.6]$ and zero else. To the right we see the function $f(x) = x\sin(x)$ and an approximation with $n = 20$ neural networks.

Proof. Pick some interval $[a, b]$ in $[0, 1]$, then look at the function

$$f(x) = \sigma(n(x-a))) - \sigma(n(x-b)) \ .$$

for large n. This function approximates the function which is 1 on $[a, b]$ and 0 else. Any function that is piece-wise constant can now be approximated by sums of such neural networks. We can therefore approximate any function. A more explicit approximation with neural networks is $g(x) = \sum_{k=1}^{n}[f(\frac{k}{n}) - f(\frac{k-1}{n})]\sigma(n(x-\frac{k}{n}))$. $\square$

33.2. Remarks:
1) This is one of many approximation theorems. **Taylor series** allow to approximate smooth functions by polynomials. **Fourier series** allow to approximate continuous functions by sums of trigonometric functions. In artificial intelligence, the sigmoid function is nice because as we have seen, the derivative can be written again in terms of the sigmoid function. This allows fast computation of **neural networks**. In order to speed this up even more, one sometimes uses piecewise linear functions and discrete derivatives. Speed is important. Llama 3 has 70 billion parameters, GPT 4 has 1.7 trillion parameters. 2) If you should look this theorem, you will see other notation. The sum of these functions can be written as $\sigma(\vec{a}x + \vec{b}) \cdot \vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are vectors, arrays of numbers and the activation is applied to each entry. In higher dimensions, one uses matrices.
3) What does "approximation" means? One way to give a distance between two functions is to compute $\int_0^1 |f(x) - g(x)|\, dx$ meaning to compute the absolute area between the graphs. If this is small, then the graphs are close.

Random Functions

33.3. Let us ask an AI teacher to automatically build worksheets or exam problems as well as solutions. In order to generate problems, we first must build **random functions**. When asked "give me an example of a function", the system should generate functions of some complexity:

Definition: A **basic function** is a function from the 10 functions $\{\sin, \cos, \log, \exp, \tan, \text{sqt}, \text{pow}, \text{inv}, \text{sca}, \text{tra}\ \}$.

33.4. Here $\text{sqt}(x) = \sqrt{x}$ and $\text{inv}(x) = 1/x^k$ for a random integer k between -1 and -3, $\text{pow}(x) = x^k$ for a random integer k between 2 and 5. $\text{sca}(x) = kx$ is a scalar multiplication for a random nonzero integer k between -3 and 3 and $\text{tra}(x) = x + k$ translates for a random integer k between -4 and 4.

33.5. Second, we use addition, subtraction multiplication, division and composition to build more complicated functions:

Definition: A **basic operation** is an operation from the list $\{f \circ g, f + g, f * g, f/g, f - g\ \}$.

33.6. The operation x^y is not included because it is equivalent to $\exp(x \log(y)) = \exp \circ (x \cdot \log)$. We can now build functions of various complexities:

Definition: A **random function** of complexity n is obtained by taking n random basic functions $f_1, \dots, f_n$, and n random basic operators $\oplus_1, \dots, \oplus_n$ and forming $f_n \oplus_n f_{n-1} \oplus_{n-1} \cdots \oplus_2 f_1 \oplus_1 f_0$ where $f_0(x) = x$ and where we start forming the function from the right.

Visitor: "Give me an easy function": Sofia looks for a function of complexity one: like $x\tan(x)$, or $x+\log(x)$, or $-3x^2$, or $x/(x-3)$.

Visitor: "Give me a function": Sofia returns a random function of complexity two: $x\sin(x)-\tan(x)$, or $-e^{\sqrt{x}}+\sqrt{x}$ or $x\sin(x)/\log(x)$ or $\tan(x)/x^4$.

Visitor: "Give me a difficult function": Sofia builds a random function of complexity four like $x^4e^{-\cos(x)}\cos(x)+\tan(x)$, or $x-\sqrt{x}-e^x+\log(x)+\cos(x)$, or $(1+x)(x\cot(x)-\log(x))/x^2$, or $(-x+\sin(x+3)-3)\csc(x)$

33.7. Now, we can build a random calculus problem. To give you an idea, here are some templates for integration problems:

Definition: A **random integration problem** of complexity n is a sentence from the sentence list { "Integrate $f(x)=F(x)$", "Find the anti derivative of $F(x)$", "What is the integral of $f(x)=F(x)$?", "You know the derivative of a function is $f'(x)=F(x)$. Find $f(x)$." }, where F is a random function of complexity n.

Visitor "Give me a differentiation problem". **Sofia:** Differentiate $f(x)=x\sin(x)-\frac{1}{x^2}$. The answer is $\frac{2}{x^3}+\sin(x)+x\cos(x)$.

Visitor: "Give me a difficult integration problem". **Sofia**: Find f if $f'(x)=\frac{1}{x}+\left(3\sin^2(x)+\sin(\sin(x))\right)\cos(x)$. The answer is $\log(x)+\sin^3(x)-\cos(\sin(x))$.

Visitor: "Give me an easy extremization problem". **Sofia**: Find the extrema of $f(x)=x/\log(x)$. The answer is $x=e$.

Visitor: "Give me an extremization problem". **Sofia**: Find the maxima and minima of $f(x)=x-x^4+\log(x)$. The extrema are

$$-\frac{\sqrt{\left(9+\sqrt{3153}\right)^{2/3}-8\sqrt[3]{6}}+\sqrt{8\sqrt[3]{6}-\left(9+\sqrt{3153}\right)^{2/3}\left(1+6\sqrt{\frac{2}{9+\sqrt{3153}-8\sqrt[3]{6\left(9+\sqrt{3153}\right)}}}\right)}}{2 2^{5/6}\sqrt[3]{3}\sqrt[6]{9+\sqrt{3153}}}.$$

As we will see in class, AI content can generate homework or worksheets quickly. Automated problem generation is the "fast food" of teaching and usually not healthy.

Homework

Problem 33.1: Illustrate the universal approximation theorem to approximate the function which is 3 on $[0.1, 0.2]$ and -2 on $[0.5, 0.6]$. Include a plot of your function.

Problem 33.2: We build a differentiation problem by combining log and sin and exp. Differentiate all of the 6 combinations $\log(\sin(\exp(x)))$, $\log(\exp(\sin(x)))$,$\exp(\log(\sin(x)))$, $\exp(\sin(\log(x)))$, $\sin(\log(\exp(x)))$ and $\sin(\exp(\log(x)))$.

Problem 33.3: Four of the 6 combinations of log and sin and exp can be integrated as elementary functions.
a) Find all these cases
b) Do these integrals.

Problem 33.4: From the 10 functions f and 10 functions g and 5 operations, we can build 500 functions. Some can not be integrated. An example is $\exp(\sin(x))$. Find 4 more which can not be integrated by you now by any computer algebra system.

Problem 33.5: We are getting to the end of the course. Ask your favorite AI to write a short exam with 5 problems the entire exam should not be longer than 1 page and doable in an hour.
Problem 1. should be about continuity Problem 2. should be about limits Problem 3. should be about differentiation Problem 4. should be about integration Problem 5. Problem should be about extrema.

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Unit 34: Music

Music is a function

34.1. A piece of music is a pair of **functions** $f(t)$ and $g(t)$. They tell the displacement of the membrane of the left and right loudspeakers. This motion produces sound waves that reach your ear. In Mathematica, we can play a function by replacing "Plot" with "Play". For example:

```
Play[ Sin[2Pi 1000 x^2],{x,0,10}]
```

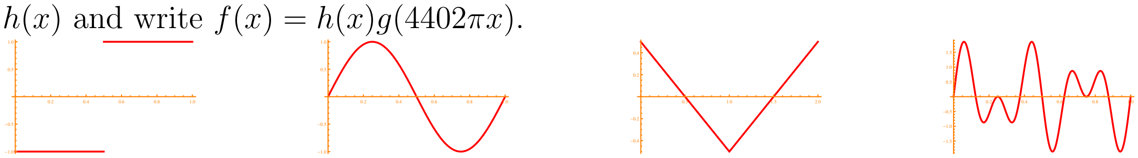
34.2. While the function f contains all the information about the music piece, the computer needs to store this in the form of **data**. The ".WAV" file for example contains 44100 sample readings per second. We can not hear higher than 20'000 KHz. A theorem of Nyquist-Shannon assures that 44.1KHz is good. To get from the sampled values $f(k)$ a function, the **Whittaker-Shannon interpolation formula**

$$f(t) = \sum_{k=1}^{n} f(k)\text{sinc}(t-k)$$

can be used. It involves the **sinc** function $\text{sinc}(x) = \sin(x)/x$.

The wave form and hull

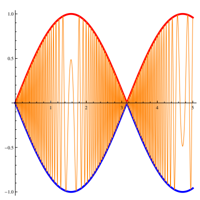
34.3. Periodic signals can serve as **building blocks** of sound. Assume $g(x)$ is a 2π-periodic function, we can generate a sound of 440 Hertz when playing the function $f(x) = g(440 \cdot 2\pi x)$. If the function does not have a smaller period, then we hear the A **tone**. It is a tone with 440 Hertz. We can **modulate** this sound with a **hull function** $h(x)$ and write $f(x) = h(x)g(4402\pi x)$.



34.4. The periodic function g is called a **wave form**. It gives a **timbre** of a sound. Music instruments can be modeled using parameters like **attack**, **vibrato**, **coloration**, **noise**, **echo** or **reverberation**.

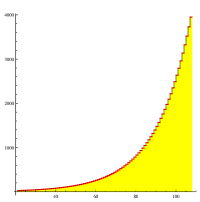
Definition: The **hull function** $h(x)$ is an interpolation of successive local maxima of f.

34.5. For the function $f(x) = \sin(100x)$ for example, the hull function is $h(x) = 1$ For $f(x) = \sin(x)\sin(100x)$ the hull function is $h(x) = |\sin(x)|$. With slight abuse of notation, we sometimes just say $\sin(x)$ is the hull function as the function is sandwiched by the envelopes $\sin(x)$ and $-\sin(x)$.



We can not hear the actual function $f(x)$ because it changes too fast. We can not follow the individual vibrations. But we can hear the hull function. as well as **large scale amplitude** changes like **creshendi** or **diminuendi** or a **vibrato**. When playing frequencies that are close, we can notice **interference**, the sound analogue of **Moiré patterns** in optics.

The scale



34.6. Western music uses a discrete set of frequencies. This scale is based on the **exponential function**. The frequency f is an exponential function of the scale s. On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

Definition: A frequency f has the **Midi number** $s = 69 + 12 \cdot \log_2(f/440)$. The **piano scale function** or **midi function** gives back $f(s) = 440 \cdot 2^{(s-69)/12}$.

34.7. The Midi tone $s = 100$ for example is a sound of $f = 2637.02$ Hertz (oscillations per second).

The **piano scale function** $f(s) = 440 \cdot 2^{(s-69)/12}$ is an exponential function $f(s) = be^{as}$ which satisfies $f(s + 12) = 2f(s)$.

```
midifrequency[m_] := N[440 2^((m - 69)/12)]
```

34.8. A classical piano has 88 keys which scale from 21 to 108. The frequency ranges from $f = 27.5Hz$, the sub-contra-octave A, to the highest $f = 4186.01Hz$, the 5-line octave C.

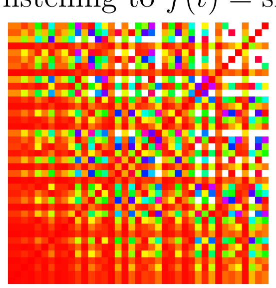
34.9. Filters: a function can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can so hear melodies, filter out part of the music and hum it.
Pitch and autotune: it is possible to filter out frequencies and adapt their frequency. The popular filter **autotune** moves the frequencies around correcting wrong singing. If 440 Hertz (A) and 523.2 Hertz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4\cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4\cos(2\pi 523.2x)$. **Rip and remix**: if f and g are two songs, we can build the average $(f+g)/2$. A composer does this using **tracks**. Different instruments are recorded independently and then mixed together. A guitar $g(t)$, a voice $v(t)$ and a piano $p(t)$ together can form $f(t) = ag(t)+bv(t)+c(p(t)$ with suitably chosen constants a, b, c. **Reverberate and echo**: if f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x+h) - f(x)]/h$. For small h, like $h = 1/1000$ the song does not change much because hearing $\sin(kx)$ or $\cos(kx)$ produces the same song. However, for larger h, one can get **reverberate** or **echo** effects.

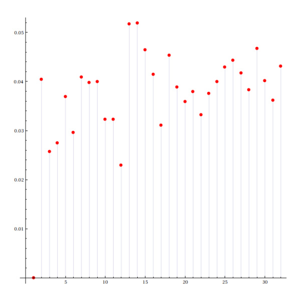
34.10. Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**. A **Diophantine problem** is the question how well a frequency can be approximated by rationals. Why is the **chromatic scale** based on $2^{1/12}$ so effective? **Indian music** for example uses **micro-tones** and a scale of 22. The 12-tone scale has the property that many powers $2^{k/12}$ are close to rational numbers. This can be quantified with the **scale fitness**

$$M(n) = \sum_{k=1}^{n} \min_{\mathrm{p,q}} |2^{k/n} - \frac{p}{q}| G(p/q)$$

where $G(n/m)$ is Euler's **gradus suavitatis** (="degree of sweetness") defined as $G(n/m) = 1 + \sum_{p|n*m}(p-1)$ in which the sum runs over all prime factors p of $n*m$. For example $G(3/4) = 1 + (2-1) + (2-1) + (3-1)$ because $3*4 = 12 = 2*2*3$.

34.11. The figure below illustrates why the 12-tone scale minimizes $M(n)$. We could also replace the concept of octave. Stockhausen experimented with replacing 2 with 5 and used the **Stockhausen scale** $5^{k/25}$. It is $f(t) = \sin(2\pi t 440 \cdot 5^{[t]/25})$, where $[t]$ is the largest integer smaller than t. The familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 440 \cdot 2^{[t]/12})$.





The perfect fifth 3/2 has the gradus suavitatis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth 4/3 for which $1 + E(12) = 1 + (2-1)(3-1)$. You can listen to the perfect fifth $f(x) = \sin(1000x) + \sin(1500x)$ or the perfect fourth $\sin(1000x) + \sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x) + \sin(1333x) + \sin(1500x) + \sin(2000x)$.

Homework

Problem 34.1: Modulation. Draw and play the following function

$$f(x) = \cos(4000x) - \cos(4011x)$$

for three seconds. You can use Wolfram alpha to do that Compare it with when playing

$$f(x) = \cos(4000x) + Cos[2000x]$$

Here is how to play a function with Mathematica:

```
Play[Cos[x]  Sin[Exp[2  x]]/x,  {x,  0,9}]
```

Problem 34.2: Amplitude modulation (AM): If you listen to $f(x) = |\cos(x^2)|\sin(1000x)$ you hear an amplitude change. Draw the hull function or listen to it and count how many increases in amplitudes to you hear in 10 seconds.

Problem 34.3: Other tonal scales, Midi number: As a creative musician, we create our own tonal scale. You decide to take the 8'th root of 3 as your basic frequency change from one tone to the next.
a) After how many tonal steps has the frequency f tripled?
b) Build the midi function and then write down the inverse for your tonal scale.

Problem 34.4: a) What is the frequency of the Midi number $s = 22$?
b) Which midi number belongs to the frequency $f = 2060 Herz$?

Problem 34.5: Log and Exp rules. a) Give an example showing $a^{(b^c)} \neq (a^b)^c$.
b) Simplify $\ln(e^{100} \cdot e^{50})$.
c) Which of the following expressions are integers? $\ln(e^{\ln(e^{10})})$, $\ln(e+e)$, $\ln(e)+\ln(e)$, $\ln(e) + \ln(e^2)$, $\ln(e^{\ln(e)})$.
d) Which of the following expressions are true $\ln(a + b) = \ln(a) + \ln(b)$, $\ln(ab) = \ln(a) \cdot \ln(b)$, $\ln(ab) = \ln(a) + \ln(b)$, $\ln(a^b) = \ln(a)\ln(b)$? e) Which of the following expressions are true $e^{a+b} = e^a + e^b$, $e^{ab} = e^a e^b$, $e^{a+b} = e^a e^b$, $(e^b)^c = e^{bc}$, $e^{(b^c)} = e^{bc}$?

INTRODUCTION TO CALCULUS

MATH 1A

Unit 35: Review (since last review)

Substitution

Substitution replaces $\int f(x)\, dx$ with $\int g(u)\, du$ with $u = u(x), du = u'(x)dx$. Cases:

A) The integral of $f(x) = g(u(x))u'(x)$, is $G(u(x))$ where G is the anti-derivative of g. B) $\int f(ax + b)\, dx = F(ax + b)/a$, where F is the anti-derivative of f.

Examples:
A) $\int \sin(x^5)x^4\, dx = \int \sin(u)\, du/5 = -\cos(u)/5 + C = -\cos(x^5)/5 + C$.
B) $\int \log(x + 7)\, dx = \int \log(u)\, du = (u \log(u) - u) + C = (x + 7) \log(x + 7) - (x + 7) + C$.

Integration by parts

Use LIATE to decide what to differentiate.

A) Direct (use this if it can be done in one step)

$$\int \mathrm{x}\ \sin(x)\ dx = \mathrm{x}\ (-\cos(x)) - \int 1\ (-\cos(x))\ dx = -x\cos(x) + \sin(x) + C\ dx\ .$$

B) Tic-Tac-Toe: for situations with an x^n in front. Example $x^2 \sin(x)$:

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	$\oplus$
2	$-\sin(x)$	$\ominus$
0	$\cos(x)$	$\oplus$

The anti-derivative is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\ .$$

C) Merry go round: Example $I = \int \sin(x)e^x\, dx$. Use parts twice and solve for I.

Partial fractions

A) Either make a common denominator on the right hand side

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$$

and compare coefficients $1 = Ax - Ab + Bx - Ba$ to get $A + B = 0, Ab - Ba = 1$ and solve for A, B.

B) Use the residue method: $f(x) = p(x)/(x-a)(x-b)$ with $a \neq b$, the coefficients A, B in

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

can be obtained as $A = p(a)/(a-b)$ and $B = p(b)/(b-a)$.

Examples:
A) $\int \frac{1}{(x+1)(x+2)}\, dx = \int \frac{A}{x+1}\, dx + \int \frac{B}{x+2}\, dx$. Find A, B by multiplying out and comparing coefficients in the nominator.

B) Better: get A by plugging in $x = -2$ after multiplying with $x - 2$. Get B by plugging in $x = -1$ after multiplying with $x - 1$.

Checklists:

Integral techniques to consider

Try to crack the integral in the following order:

- [ ] Know the integral
- [ ] Substitution
- [ ] Integration by parts
- [ ] Partial fractions

Especially cool parts:

- [ ] Tic-Tac-Toe for integration by parts
- [ ] Hospital Method for partial fractions
- [ ] Merry go round method for parts

Integrals to know well

- [ ] $\sin(x)$
- [ ] $\cos(x)$
- [ ] $\tan(x)$
- [ ] $\log(x)$
- [ ] $\exp(x)$
- [ ] $1/x$

- $1/x^n$
- x^n
- $\sqrt{x}$
- $1/\cos^2(x)$
- $1/\sin^2(x)$
- $1/(1+x^2)$
- $1/(1-x^2)$
- $1/\sqrt{1-x^2}$

Applications

- **Average rate of change:** $[f(x+h)-f(x)]/h$
- **Derivative:** Limit of differences $[f(x+h)-f(x)]/h$ for $h \to 0$
- **Integral:** Limit of Riemann sums $S_n = [f(x_0)+f(x_1)+\cdots+f((n-1)x_k)]\Delta x$.
- **Newton step:** $T(x) = x - f(x)/f'(x)$.
- **Velocity:** Derivative of the position.
- **Acceleration:** Derivative of the velocity.
- **Concavity:** measured by $f''(x)$.
- **Marginal cost:** the derivative F' of the total cost F.
- **Average cost:** F/x where F is the total cost.
- **Probability distribution function:** non-negative, total $\int f(x)dx = 1$.
- **Cumulative distribution function:** anti-derivative of the PDF.
- **Expectation:** $\int xf(x)\,dx$, where f is the probability density function.
- **Moments:** $\int x^n f(x)\,dx$, where f is the probability density function.

Terminology

- **Operations research:** find extrema, critical points, derivative tests
- **Crisis management:** critical points and catastrophes.
- **Economics:** average, mar cost, marginal cost and total cost. Strawberry theorem.
- **Computer science:** sigmoid function $\sigma(x) = 1/(1+e^{-x})$, neural networks.
- **Statistics:** PDF, CDF, expectation, variance.
- **Distributions:** normal, geometric, exponential, Cauchy, arcsin, logistic.
- **Treasure hunting:** Bisection method, Newton method $T(x) = x - f(x)/f'(x)$.
- **Artificial intelligence:** sigmoid function, neural net functions, attention.
- **Music:** hull function, piano function.
- **Gastronomy:** wobbling table, bottle calibration.

Other notions

- **Hull function:** $\sin(x)\sin(10000x)$ has hull $|\sin(x)|$
- **Catastrophe:** A parameter c at which a local minimum disappears.
- **Artificial intelligence:** Sigmoid σ, neural network $c\sigma(ax+b)$
- **Entropy:** $-\int f(x)\log(f(x))\,dx$.

- **Bottles:** How to calibrate bottles. The calibration formula.
- **Wobbly chair:** One can turn a chair on any lawn to stop it from wobbling.
- **Derivative rule:** The hit: "low d high take high d low, cross the line and square the low"
- **Midi function:** $f(m) = 440 * 2^{(m-69)/12}$.

Core concepts:

- **Algebra:** Algebra, Power, Log
- **Fundamental:** The fundamental theorem of calculus
- **Trigonometry:** Fundamental theorem of trigonometry
- **Intermediate:** Intermediate value theorem
- **Mean value:** Mean value theorem
- **Extrema:** First derivative test
- **Extrema:** Second derivative test
- **Derivatives:** slope rate of change
- **Integrals:** area, volume
- **Limits:** Definitions and Hospital!
- **Limits:** Squeeze theorem
- **Continuity:** know the enemies of continuity
- **Numerics:** Riemann sums
- **Rules:** Differentiation and integration rules.
- **Methods:** Integration by parts, Substitution, Partial fraction.

People:

- **Newton, Leibniz**: Fundamental theorem, Newton method
- **Hospital, Bernoulli**: Hospital method
- **Riemann**: Riemann sum
- **Bolzano**: Intermediate and mean value theorem
- **Rolle**: Rolle's theorem
- **Fermat**: Fermat principle
- **Gini**: Gini Coefficient
- **Thom, Zeeman**: Catastrophes
- **Zeno**: Zeno paradox
- **Knuth**: Knuth notation
- **Bolt, Neyad**: Sports
- **Gibbs, Bolzmann, Shannon, Helmholz**: Entropy, Energy
- **Polya, Tao:** How to solve.
- **Whittaker-Shannon:** Interpolation.
- **Euler:** Gradus Suavitatis.
- **Gonnick:** Cartoons

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